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Determinant of Laplacians on Heisenberg manifolds

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Abstract

We give an integral representation of the *zeta-regularized determinant* of Laplacians on three-dimensional Heisenberg manifolds, and study a behavior of the values when we deform the uniform discrete subgroups. Heisenberg manifolds are the total space of a fiber bundle with a torus as the base space and a circle as a typical fiber, then the deformation of the uniform discrete subgroups means that the "radius" of the fiber goes to zero. We explain the lines of the calculations precisely for three-dimensional cases and state the corresponding results for five-dimensional Heisenberg manifolds. We see that the values themselves are of the product form with a factor which is that of the flat torus. So in the last half of this paper we derive general formulas of the zeta-regularized determinant for product type manifolds of two Riemannian manifolds, discuss the formulas for flat tori and explain a relation of the formula for the two-dimensional flat torus and the *Kronecker's second limit formula*.

Keywords: Heisenberg group; Zeta-regularized determinant; Laplacian; Heat kernel; Kronecker's second limit formula; Modified Bessel function; Poisson's summation formula

1. Introduction

In the conformal field theory, especially in the string theory an infinite dimensional analog of the determinant for elliptic operators appears and it is considered in the framework of the analytic continuation method of the spectral zeta-function of elliptic operators. In the string theory it seems to be most interesting to calculate the explicit values for compact Riemannian surfaces and also from the mathematical point of view the quantity relates with various aspects of special functions already in such two-dimensional cases. As for the

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values for the spheres there are relations with the values of the Riemann zeta-functions at the negative integers, for the two-dimensional flat torus it is expressed by using the famous formula, the so-called *Kronecker's second limit formula*, and for the cases of compact surfaces with constant negative curvature it was calculated by using Selberg's trace formula and deep properties of modified Bessel functions [4,5,7,9,14]. Thus for all two-dimensional cases we already know the values of the zeta-regularized determinant of the Laplacians or relations with other quantities. A purpose of this note is to give an integral representation of the zeta-regularized determinant for three-dimensional Heisenberg manifolds (see [11] for a class of three-dimensional lens spaces) and state the corresponding results for five-dimensional Heisenberg manifolds. It will be possible to give similar expressions for higher dimensional cases, but we restrict ourselves to these two cases and also we restrict ourselves to deal with a certain kind of uniform discrete subgroups of the Heisenberg group, since even in these cases they contain all necessary features for determining the values for any cases and make us the calculations to be simple. Then in the last half of this paper we derive a general formula of the zeta-regularized determinant for manifolds of the product form of two Riemannian manifolds and the formulas for flat tori of two, three and four dimensions.

In Section 2 we gather up the basic data of the spectrum of the three-dimensional Heisenberg group and Heisenberg manifolds. In Section 3 first, we explain the zeta-regularized determinant of the Laplacian and give a calculation for the three-dimensional Heisenberg manifolds based on an integral representation of the spectral zeta-function (=the Mellin transformation of the trace of heat kernel divided by a Gamma function). In Section 4 as an application of an integral representation of the spectral zeta-function given in Section 3 we give expressions of the all coefficients of the asymptotic expansion of the heat kernel for the three-dimensional Heisenberg manifolds. In Section 5 we state the corresponding results of Sections 3 and 4 for five-dimensional Heisenberg manifolds. In Section 6 we give a general expression of the zeta-regularized determinant of the Laplacian on the product of two Riemannian manifolds. In Section 7 we give a precise form of the formula derived in Section 6 for a product type manifold with S^1. Finally in Section 8 we give such formulas for two-, three- and four-dimensional flat tori and explain a relation of the formula for two-dimensional flat torus and the Kronecker's second limit formula.

2. Spectrum of three-dimensional Heisenberg manifolds

Let H_3 be the three-dimensional Heisenberg group:

$$H_3 = \left\{ g = g(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}. \tag{2.1}$$

The Lie algebra is

$$\mathfrak{h}_3 = \left\{ X = X(x, y, z) = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}. \tag{2.2}$$

It is decomposed into a direct sum in the form of

$$\mathfrak{h}_3 = \mathfrak{g}_+ \oplus \mathfrak{g}_- \oplus \mathfrak{z}, \tag{2.3}$$

where

$$\mathfrak{g}_+ = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\}, \qquad \mathfrak{g}_- = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| y \in \mathbb{R} \right\}$$

and

$$\mathfrak{z} = \text{the center} = \left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| z \in \mathbb{R} \right\}.$$

We identify $\mathfrak{h}_3$ and H_3 through the exponential map. Then the group multiplication of two elements $X = X(x, y, z) \in \mathfrak{h}_3$ and $\tilde{X} = \tilde{X}(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathfrak{h}_3$ is given by

$$Y = X * \tilde{X} = X + \tilde{X} + \tfrac{1}{2}[X, \tilde{X}], \text{ that is, } \exp(Y) = \exp X \exp \tilde{X}. \tag{2.4}$$

Left invariant Riemannian metrics are determined by its restriction to the tangent space at the identity element ($\cong$ its Lie algebra), and among the left invariant Riemannian metrics we only consider such a metric that

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Z_0 = z_0 \cdot Z = z_0 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are being an orthonormal basis. Also we only consider uniform discrete subgroups Γ_ℓ of the following form:

$$\Gamma_\ell = \left\{ \begin{pmatrix} 1 & m & \frac{k}{2\ell} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \middle| m, n, k \in \mathbb{Z} \right\}, \tag{2.5}$$

where ℓ is a positive integer. These choices do not loose the essential features in treating with the spectrum of the Laplacian on the quotient space, H_3/Γ_ℓ, the so-called Heisenberg manifolds, since the spectrum is given more or less in a similar form [6] (we state them later).

Then the inverse image of Γ_ℓ by the exponential map is

$$\exp^{-1}(\Gamma_\ell) = \left\{ \left(\begin{array}{ccc} 0 & m & \dfrac{k}{2\ell} \\ 0 & 0 & n \\ 0 & 0 & 0 \end{array} \right) \middle| m, n, k \in \mathbb{Z} \right\}, \tag{2.6}$$

which is a direct sum of two uniform lattices Γ_B in $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ and $\Gamma_V(\ell)$ in $\mathfrak{z}$ such that

$$\Gamma_B = \left\{ \left(\begin{array}{ccc} 0 & m & 0 \\ 0 & 0 & n \\ 0 & 0 & 0 \end{array} \right) \middle| m, n \in \mathbb{Z} \right\} \tag{2.7}$$

and

$$\Gamma_V(\ell) = \left\{ \left(\begin{array}{ccc} 0 & 0 & \dfrac{k}{2\ell} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| k \in \mathbb{Z} \right\}. \tag{2.8}$$

The kernel $K_t(g, \tilde{g})$ of $\mathrm{e}^{-t\Delta}$, the heat kernel on the Heisenberg group H_3, is given by

$$\begin{aligned} K_t(g, \tilde{g}) &= K_t(x, y, z; \tilde{x}, \tilde{y}, \tilde{z}) = (2\pi)^{-2} \int_{-\infty}^{+\infty} \mathrm{e}^{\sqrt{-1}\eta\{\tilde{z}-z+(1/2)(\tilde{x}y-x\tilde{y})\}}\, \mathrm{e}^{-t|\eta|^2} \\ &\quad \times \frac{|\eta|}{2\sinh t|\eta|}\, \mathrm{e}^{-(|\eta|/4)(\cosh t|\eta|/\sinh t|\eta|)\{(x-\tilde{x})^2+(y-\tilde{y})^2\}}\, \mathrm{d}\eta, \end{aligned} \tag{2.9}$$

where we regard $\eta \in \mathfrak{z}^* = \mathbb{R}Z^* \cong \mathbb{R}$ and $g = xX_1 + yY_1 + zZ$, $\tilde{g} = \tilde{x}X_1 + \tilde{y}Y_1 + \tilde{z}Z \in \mathfrak{h}_3 \cong H_3$ through the identification by the exponential map $\exp : \mathfrak{h}_3 \to H_3$.

Then the heat kernel $k_{H_3/\Gamma_\ell}(t; [g], [\tilde{g}])$ on the Heisenberg manifold H_3/Γ_ℓ is expressed by making use of the heat kernel $K_t(g, \tilde{g})$ on the whole group, because of the invariance $K_t(\gamma \cdot g, \gamma \cdot \tilde{g}) = K_t(g, \tilde{g})$, $\gamma \in H_3$:

$$k_{H_3/\Gamma_\ell}(t; [g], [\tilde{g}]) = \sum_{\gamma \in \Gamma_\ell} K_t(\gamma \cdot g, \tilde{g}). \tag{2.10}$$

Its trace is calculated in the following theorem.

Theorem 2.1 (Furutani [8]).

$$\begin{aligned} &\int_{H_3/\Gamma_\ell} k_{H_3/\Gamma_\ell}(t; [g], [g])\, \mathrm{d}g \\ &\quad = \sum_{\gamma \in \Gamma_\ell} \int_{F_\ell} K_t(\gamma \cdot g, g)\, \mathrm{d}g \\ &\quad = \sum_{\mu \in \Gamma_V(\ell)^*, \mu \neq 0} \sum_{m=0}^{\infty} \mathrm{Vol}\left(\frac{\mathfrak{g}_+ \oplus \mathfrak{g}_-}{\Gamma_B} \right) \|\mu\|\, \mathrm{e}^{-t\{4\pi^2\|\mu\|^2 + 2\pi(2m+1)\|\mu\|\}} + \sum_{\nu \in \Gamma_B^*} \mathrm{e}^{-4\pi^2 t \|\nu\|^2}. \end{aligned}$$

Here F_ℓ denotes a fundamental domain of the uniform discrete subgroup Γ_ℓ and Γ_B^* and $\Gamma_V(\ell)^*$ are dual lattices, i.e.,

$$\begin{aligned}\Gamma_B^* &= \{\nu \in (\mathfrak{g}_+ \oplus \mathfrak{g}_-)^* | \nu(\gamma) \in \mathbb{Z} \text{ for any } \gamma \in \Gamma_B\}\\ &\cong \{\nu = \nu_1 X_1^* + \nu_2 Y_1^* | \nu_i \mathbb{Z}\}, \quad \{X_1^*, Y_1^*\} \text{ are dual basis of } \mathfrak{g}_+ \oplus \mathfrak{g}_-,\end{aligned}$$

$$\begin{aligned}\Gamma_V(\ell)^* &= \{\mu \in (\mathfrak{z})^* | \mu(\gamma) \in \mathbb{Z} \text{ for any } \gamma \in \Gamma_V(\ell)\}\\ &\cong \{\mu = 2\ell \cdot k Z_1^* | k \in \mathbb{Z}\}, \quad \|\mu\| = 2\ell |k|.\end{aligned}$$

The spectrum of the Laplacian on H_3/Γ_ℓ is the union of eigenvalues of two types (a) and (b).

Proposition 2.2 (Gordon and Wilson [6]).

(a) $4\pi^2\|\mu\|^2 + 2\pi(2m+1)\|\mu\|, \quad \mu = 2\ell \cdot k Z_1^* \in \Gamma_V(\ell)^*$
with the multiplicity equal to

$$\mathrm{Vol}\left(\frac{\mathfrak{g}_+ \oplus \mathfrak{g}_-}{\Gamma_B}\right)\|\mu\| = 1 \cdot 2\ell \cdot |k|, \quad k \in \mathbb{Z}.$$

(b) $4\pi^2\|\nu\|^2 = 4\pi^2(\nu_1^2 + \nu_2^2), \quad \nu = \nu_1 X_1^* + \nu_2 Y_1^* \in \Gamma_B^*.$

From the exact sequence

$$0 \to \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{R} \right\} \to H_3 \to \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} \to 0,$$

we know that the Heisenberg manifold is the total space of a principal bundle with the structure group

$$U(1) \cong \frac{\mathbb{R}}{\{2\ell \cdot k | k \in \mathbb{Z}\}}$$

and a torus

$$T^2 \cong \frac{\mathfrak{g}_+ \oplus \mathfrak{g}_-}{\Gamma_B}$$

as the base manifold. We denote the projection map $H_3/\Gamma_\ell \to T^2$ by ρ.

Proposition 2.3. *The fibers of the bundle* $\rho : H_3/\Gamma_\ell \to T^2$ *are totally geodesic and so the map* $\rho^* : C^\infty(T^2) \to C^\infty(H_3/\Gamma_\ell)$ *commutes with the action of Laplacians on the each space*.

By this property the eigenvalues of type (b) are those coming from the base space T^2 through the map ρ.

3. Zeta-regularized determinant

Let $\mathbf{M}$ be an n-dimensional closed Riemannian manifold with the Laplacian $\Delta_{\mathbf{M}}$, and denote the heat kernel by $k_{\mathbf{M}}(t; x, y)$:

$$\int_{\mathbf{M}} k_{\mathbf{M}}(t; x, y) f(y)\, \mathrm{d}y = \mathrm{e}^{-t\Delta_{\mathbf{M}}}(f)(x).$$

Then the Mellin transformation of the trace of the heat kernel,

$$\frac{1}{\Gamma(s)} \int_0^\infty \left(\int_{\mathbf{M}} k_{\mathbf{M}}(t; x, x)\, \mathrm{d}x - 1 \right) t^{s-1}\, \mathrm{d}t = \sum_{\lambda \neq 0} \frac{m_\lambda}{\lambda^s},$$

m_λ is the multiplicity of the eigenvalue λ,

is meromorphically continued to the whole complex plane with only poles of order 1 (at most) at $s = n/2,\ n/2 - 1, \ldots$, especially at $s = 0$ it is holomorphic. We put this function as $\zeta_{\mathbf{M}}(s)$ and call the spectral zeta-function of the Riemannian manifold $\mathbf{M}$. Then we can regard the value

$$\mathrm{e}^{-\zeta'_{\mathbf{M}}(0)}$$

as a *determinant* (=product of non-zero eigenvalues) of the Laplacian $\Delta_{\mathbf{M}}$ acting on the space of functions orthogonal to the space of constant functions [7,12,13] and call it as zeta-regularized determinant of the Laplacian. We denote it by $\mathrm{Det}\, \Delta_{\mathbf{M}}$. Of course it can be defined in a same way for more general elliptic operators.

In our case of the three-dimensional Heisenberg manifold $\mathbf{M}(\ell) = H_3/\Gamma_\ell$ we put

$$\sum_{\mu \in \Gamma_V(\ell)^*, \mu \neq 0} \sum_{m=0}^{\infty} \left(\mathrm{Vol}\left(\frac{\mathfrak{g}_+ \oplus \mathfrak{g}_-}{\Gamma_B} \right) |\mu| \right) \mathrm{e}^{-t\{4\pi^2 \|\mu\|^2 + 2\pi(2m+1)\|\mu\|\}} + \sum_{\nu \in \Gamma_B^*} \mathrm{e}^{-4\pi^2 t \|\nu\|^2},$$
$$Z_{\mathbf{M}(\ell)}(t) = Z_V(t) + Z_{T^2}(t),$$

then the second term is the trace of the heat kernel of the flat torus $T^2 \cong (\mathfrak{g}_+ \oplus \mathfrak{g}_-)/\Gamma_B$, and so

$$\begin{aligned} \zeta_{\mathbf{M}(\ell)}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty (Z_{\mathbf{M}(\ell)}(t) - 1) t^{s-1}\, \mathrm{d}t = \frac{1}{\Gamma(s)} \int_0^\infty Z_V(t) t^{s-1}\, \mathrm{d}t \\ &\quad + \frac{1}{\Gamma(s)} \int_0^\infty (Z_{T^2}(t) - 1) t^{s-1}\, \mathrm{d}t = \zeta_V(s) + \zeta_{T^2}(s). \end{aligned}$$

Hence we have the following proposition.

Proposition 3.1.

$$\mathrm{e}^{-\zeta_{\mathbf{M}(\ell)'(0)}} = \mathrm{e}^{-\zeta'_V(0)}\, \mathrm{e}^{-\zeta'_{T^2}(0)}.$$

The value $\zeta'_{T^2}(0)$ is given by the formula called Kronecker's second limit formula.

Proposition 3.2 (Berndt et al. [3,7,12]).

$$\mathrm{Det}\,\Delta_{T^2} = \mathrm{e}^{-\zeta'_{T^2}(0)} = \mathrm{e}^{-\pi/3}\left|\prod_{k=-\infty}^{\infty}(1-\mathrm{e}^{-2\pi|k|})\right|^2.$$

We give an elementary proof of this formula and expressions of the zeta-regularized determinant for the three- and four-dimensional flat tori in Section 8.

So in this section we only consider the value $\zeta'_V(0)$. The Mellin transform of the function $Z_V(t)$ is

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^\infty Z_V(t)t^{s-1}\,\mathrm{d}t \\
&\quad = 4\ell\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{n}{(4\pi\ell)^{2s}(n^2+(n/4\pi\ell)(2m+1))^s} \\
&\quad = \frac{4\ell}{\Gamma(s)\Gamma(s-1)}\frac{1}{(4\pi\ell)^{2s}}\int_0^\infty\int_0^\infty \frac{x+y}{\mathrm{e}^{x+y}-1}\frac{x}{\mathrm{e}^{x/4\pi\ell}-\mathrm{e}^{-(x/4\pi\ell)}}\frac{(xy)^{s-2}}{x+y}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{3.1}$$

Put

$$f(x,y) = \frac{x+y}{\mathrm{e}^{x+y}-1}\frac{x}{\mathrm{e}^{x/4\pi\ell}-\mathrm{e}^{-(x/4\pi\ell)}},$$

then by the transformation

$$\begin{aligned}
&(x,y)\mapsto(u,v), \quad u = x \text{ and } v = \frac{y}{x} \text{ on the domain } x>y,\\
&(x,y)\mapsto(v,u), \quad u = y \text{ and } v = \frac{x}{y} \text{ on the domain } x<y,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty f(x,y)\frac{(xy)^{s-2}}{x+y}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_0^1\left(\int_0^\infty f(u,uv)u^{2s-4}\,\mathrm{d}u\right)\frac{v^{s-2}}{1+v}\,\mathrm{d}v + \int_0^1\left(\int_0^\infty f(uv,u)u^{2s-4}\,\mathrm{d}u\right)\frac{v^{s-2}}{1+v}\,\mathrm{d}v \\
&\quad = \int_0^1\left(\int_0^\infty (f(u,uv)+f(uv,u))u^{2s-4}\,\mathrm{d}u\right)\frac{v^{s-2}}{1+v}\,\mathrm{d}v \\
&\quad = \int_0^1\left(\int_0^\infty G(u,v)u^{2s-4}\,\mathrm{d}u\right)\frac{v^{s-2}}{1+v}\,\mathrm{d}v,
\end{aligned}$$

where we put

$$G(u,v) = f(u,uv) + f(uv,u).$$

Let $g(x) = x/(\mathrm{e}^x-1)$ and $h(x) = 2x/(\mathrm{e}^x-\mathrm{e}^{-x}) = x/\sinh x$, then

$$G(u,v) = 2\pi\ell\cdot g(u(1+v))\left(h\left(\frac{u}{4\pi\ell}\right)+h\left(\frac{uv}{4\pi\ell}\right)\right). \tag{3.2}$$

Since the functions g and h are rapidly decreasing on the positive real axis, we have the following proposition.

Proposition 3.3. *For any integers k and l*

$$\lim_{u\to\infty}\frac{\partial^{k+l}G}{\partial u^k\partial v^l}(u,v)=0$$

uniformly for $v\in[0,1]$.

Next we consider the behavior of the functions $\partial^{k+l}G/\partial u^k\partial v^l$, when $u\downarrow 0$. Let

$$g(x)=\frac{x}{\mathrm{e}^x-1}=\sum_0^\infty \alpha_k x^k,\quad |x|<2\pi \tag{3.3}$$

and

$$h(x)=\frac{2x}{\mathrm{e}^x-\mathrm{e}^{-x}}=\frac{x}{\sinh x}=\frac{\sqrt{-1}x}{\sin\sqrt{-1}x}=\sum_0^\infty(-1)^k\beta_{2k}x^{2k},\quad |x|<\pi. \tag{3.4}$$

Note that $\alpha_0=1$, $\alpha_1=-1/2$, $\alpha_2=1/12$, $\alpha_{2i+1}=0$ for $i=1,2,3,\dots$ and $\beta_0=1$, $\beta_2=1/6$, $\beta_4=7/360$. The coefficients are expressed in the following forms with Bernoulli numbers $B_{2k}=(2(2k)!/(2\pi)^{2k})\zeta(2k)$:

$$\beta_{2k}=\frac{(2^{2k}-2)B_{2k}}{(2k)!},\qquad \alpha_{2k}=\frac{B_{2k}}{(2k)!}.$$

For $|u|<\pi$ and $v\in[0,1]$, the function $G(u,v)$ is expanded as follows:

$$\begin{aligned}G(u,v)&=2\pi\ell\sum_{n=0}^\infty\left(\sum_{i+2j=n}(-1)^j\frac{\alpha_i\beta_{2j}}{(4\pi\ell)^{2j}}(1+v)^i(1+v^{2j})\right)\cdot u^n\\&=2\pi\ell\sum_{n=0}^\infty P_n(v)u^n,\end{aligned}$$

where we denote the polynomial $P_n(v)$

$$P_n(v)=\sum_{i+2j=n}(-1)^j\frac{\alpha_i\beta_{2j}}{(4\pi\ell)^{2j}}(1+v)^i(1+v^{2j}).$$

Proposition 3.4. *For* $v\in[0,1]$

$$\lim_{u\downarrow 0}\frac{\partial^{k+l}G}{\partial u^k\partial v^l}(u,v)=2\pi\ell\cdot k!\frac{\mathrm{d}^lP_k(v)}{\mathrm{d}v^l}.$$

Let $g(v,s)$ be a sufficiently many times differentiable function defined on a domain in $\mathbb{R}\times\mathbb{C}$ including $[0,1]\times\mathbf{D}$, where $\mathbf{D}=\{s\in\mathbb{C}|\mathfrak{Re}(s)>-\epsilon\}$ ($\epsilon>0$ and fixed) and g is holomorphic on the domain $\mathbf{D}$ for each fixed $t\in[0,1]$.

Proposition 3.5. *The function defined by the integral*

$$\int_0^1 g(v,s)v^{s-2}\,\mathrm{d}v$$

has the Laurent expansion at $s=0$ as

$$\int_0^1 g(v,s)v^{s-2}\,\mathrm{d}v = \frac{R_{-1}}{s} + R_0 + \mathrm{O}(s), \tag{3.5}$$

where R_{-1} and R_0 are given by the formulas:

$$R_{-1} = \frac{\partial g}{\partial v}(0,0)$$

and

$$R_0 = -\int_0^1 \frac{\partial^2 g}{\partial v^2}(v,0)\log v\,\mathrm{d}v + \frac{\partial^2 g}{\partial s\partial v}(0,0) + \frac{\partial g}{\partial v}(0,0) - g(1,0).$$

By applying this to the function of the form $g(v,s)/(1+v)$ we have the Laurent expansion of the function

$$\int_0^1 g(v,s)\frac{v^{s-2}}{1+v}\,\mathrm{d}v$$

as follows.

Corollary 3.6.

$$\begin{aligned}\int_0^1 g(v,s)\frac{v^{s-2}}{1+v}\,\mathrm{d}v &= \left(\frac{\partial g}{\partial v}(0,0) - g(0,0)\right)\frac{1}{s} - \int_0^1 \frac{\partial^2 g}{\partial v^2}(v,0)\log v\,\mathrm{d}v \\ &\quad + \int_0^1 \frac{\partial g}{\partial v}(v,0)\log v\,\mathrm{d}v + \int_0^1 \frac{g(v,0)}{1+v}\,\mathrm{d}v + \frac{\partial^2 g}{\partial s\partial v}(0,0) \\ &\quad - \frac{\partial g}{\partial s}(0,0) + \frac{\partial g}{\partial v}(0,0) - g(1,0) + 0(s).\end{aligned}$$

When we restrict the variable s in the domain $\mathfrak{Re}(s) > 3/2$, we have

$$\int_0^\infty G(u,v)u^{2s-4}\,\mathrm{d}u = \frac{1}{(2s-3)(2s-2)(2s-1)2s}\int_0^\infty \frac{\partial^4 G(u,v)}{\partial u^4}u^{2s}\,\mathrm{d}u. \tag{3.6}$$

Now by Corollary 3.6 when we put

$$\begin{aligned}&\int_0^1 \left(\int_0^\infty G(u,v)u^{2s-4}\,\mathrm{d}u\right)\frac{v^{s-2}}{1+v}\,\mathrm{d}v \\ &\quad = \frac{1}{(2s-3)(2s-2)(2s-1)2s}\int_0^1 \left(\int_0^\infty \frac{\partial^4 G(u,v)}{\partial u^4}u^{2s}\,\mathrm{d}u\right)\frac{v^{s-2}}{1+v}\,\mathrm{d}v \\ &\quad = \frac{1}{(2s-3)(2s-2)(2s-1)2s}\left\{\frac{R_{-1}}{s} + R_0 + \mathrm{O}(1)\right\},\end{aligned}$$

then we have the following proposition.

Proposition 3.7.

$$R_{-1}=0,\qquad R_0=2\int_0^\infty \frac{\partial^5 G(u,0)}{\partial v\partial u^4}\log u\,\mathrm{d}u-2\int_0^\infty \frac{\partial^4 G(u,0)}{\partial u^4}\log u\,\mathrm{d}u$$
$$=-4\pi\ell\int_0^\infty \frac{\mathrm{d}^4}{\mathrm{d}u^4}\left(h\left(\frac{u}{2}\right)^2\left(h\left(\frac{u}{4\pi\ell}\right)+1\right)\right)\log u\,\mathrm{d}u.$$

Proof. First we show

$$R_{-1}=\int_0^\infty \frac{\partial^5 G(u,0)}{\partial v\partial u^4}\,\mathrm{d}u-\int_0^\infty \frac{\partial^4 G(u,0)}{\partial u^4}\,\mathrm{d}u=\left.\frac{\partial^4 G(u,0)}{\partial v\partial u^3}\right|_0^\infty-\left.\frac{\partial^3 G(u,0)}{\partial u^3}\right|_0^\infty$$
$$=\frac{2\pi\ell}{(4\pi\ell)^2}\cdot g'(0)h''(0)-\left.\frac{2\pi\ell}{(4\pi\ell)^2}(1+v)\cdot g'(0)h''(0)\right|_{v=0}=0.$$

Next we calculate R_0:

$$R_0=-\int_0^1\left(\int_0^\infty \frac{\partial^6 G(u,v)}{\partial v^2\partial u^4}\,\mathrm{d}u\right)\log v\,\mathrm{d}v+\int_0^1\left(\int_0^\infty \frac{\partial^5 G(u,v)}{\partial v\partial u^4}\,\mathrm{d}u\right)\log v\,\mathrm{d}v$$
$$+\int_0^1\left(\int_0^\infty \frac{\partial^4 G(u,v)}{\partial u^4}\,\mathrm{d}u\right)\frac{1}{1+v}\,\mathrm{d}v+\int_0^\infty \frac{\partial^5 G(u,0)}{\partial v\partial u^4}2\log u\,\mathrm{d}u$$
$$-\int_0^\infty \frac{\partial^4 G(u,0)}{\partial u^4}2\log u\,\mathrm{d}u+\int_0^\infty \frac{\partial^5 G(u,0)}{\partial v\partial u^4}\,\mathrm{d}u-\int_0^\infty \frac{\partial^4 G(u,1)}{\partial u^4}\,\mathrm{d}u$$
$$=\int_0^1 \frac{\partial^5 G(0,v)}{\partial v^2\partial u^3}\log v\,\mathrm{d}v-\int_0^1 \frac{\partial^4 G(0,v)}{\partial v\partial u^3}\log v\,\mathrm{d}v-\int_0^1 \frac{\partial^3 G(0,v)}{\partial u^3}\frac{1}{1+v}\,\mathrm{d}v$$
$$+2\int_0^\infty \frac{\partial^5 G(u,0)}{\partial v\partial u^4}\log u\,\mathrm{d}u-2\int_0^\infty \frac{\partial^4 G(u,0)}{\partial u^4}\log u\,\mathrm{d}u-\frac{\partial^4 G(0,0)}{\partial v\partial u^3}$$
$$+\frac{\partial^3 G(0,1)}{\partial u^3}=2\pi\ell\cdot 3!\int_0^1\left(\frac{\mathrm{d}^2 P_3(v)}{\mathrm{d}v^2}-\frac{\mathrm{d}P_3(v)}{\mathrm{d}v}\right)\log v-\frac{P_3(v)}{1+v}\,\mathrm{d}v$$
$$+2\pi\ell\cdot 3!\left\{-\frac{\mathrm{d}P_3(0)}{\mathrm{d}v}+P_3(1)\right\}+2\int_0^\infty \frac{\partial^5 G(u,0)}{\partial v\partial u^4}\log u\,\mathrm{d}u$$
$$-2\int_0^\infty \frac{\partial^4 G(u,0)}{\partial u^4}\log u\,\mathrm{d}u=2\pi\ell\cdot 3!\cdot(-3)\cdot P_3(0)$$
$$+2\pi\ell\cdot 3!\cdot 3\cdot P_3(0)+2\int_0^\infty \frac{\partial^5 G(u,0)}{\partial v\partial u^4}\log u\,\mathrm{d}u-2\int_0^\infty \frac{\partial^4 G(u,0)}{\partial u^4}\log u\,\mathrm{d}u$$
$$=2\int_0^\infty \frac{\partial^5 G(u,0)}{\partial v\partial u^4}\log u\,\mathrm{d}u-2\int_0^\infty \frac{\partial^4 G(u,0)}{\partial u^4}\log u\,\mathrm{d}u$$
$$=-4\pi\ell\int_0^\infty \frac{\mathrm{d}^4}{\mathrm{d}u^4}\left(h\left(\frac{u}{2}\right)^2\left(h\left(\frac{u}{4\pi\ell}\right)+1\right)\right)\log u\,\mathrm{d}u.\qquad\square$$

Summing up we have

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^\infty Z_V(t)t^{s-1}\,\mathrm{d}t\\
&\quad=\frac{4\cdot\ell\cdot(s-1)}{\Gamma(s+1)^2}\,\frac{s^2}{(4\pi\ell)^{2s}}\,\frac{1}{(2s-3)(2s-2)(2s-1)2s}\{R_0+\mathrm{O}(s)\}=\frac{\ell R_0}{3}s+\mathrm{O}(s^2),
\end{aligned}$$

and the zeta-regularized determinant of the Laplacian on the Heisenberg manifold H_3/Γ_ℓ is given by the following formula.

Theorem 3.8.

$$\begin{aligned}
\operatorname{Det}\Delta_{H_3/\Gamma_\ell} &= \operatorname{Det}\Delta_{T^2}\,\mathrm{e}^{-\ell R_0/3}=\mathrm{e}^{-\pi/3}\left|\prod_{k=-\infty}^{\infty}(1-\mathrm{e}^{-2\pi|k|})\right|^2\\
&\quad\times\mathrm{e}^{(4\pi\ell^2/3)\int_0^\infty(\mathrm{d}^4/\mathrm{d}u^4)((u/2)/(\sinh u/2))^2((u/4\pi\ell)/(\sinh(u/4\pi\ell)+1))}\log u\,\mathrm{d}u.
\end{aligned}\tag{3.7}$$

Corollary 3.9. *When* $\ell\to\infty$, $\operatorname{Det}\Delta_{H_3/\Gamma_\ell}\to 0$.

Proof. Since

$$\begin{aligned}
&\lim_{\ell\to\infty}\int_0^\infty\frac{\mathrm{d}^4}{\mathrm{d}u^4}\left\{h\left(\frac{u}{2}\right)^2\left(h\left(\frac{u}{4\pi\ell}\right)+1\right)\right\}\log u\,\mathrm{d}u\\
&\quad=2\int_0^\infty\frac{\mathrm{d}^4}{\mathrm{d}u^4}\left\{h\left(\frac{u}{2}\right)^2\right\}\log u\,\mathrm{d}u,
\end{aligned}\tag{3.8}$$

we only determine the sign of this integral. For this purpose we decompose the integral (3.8) in the form

$$\begin{aligned}
I&=\int_0^\infty\frac{\mathrm{d}^4}{\mathrm{d}u^4}\left\{h\left(\frac{u}{2}\right)^2\right\}\log u\,\mathrm{d}u\\
&=\int_0^r\frac{\mathrm{d}^4}{\mathrm{d}u^4}\left\{h\left(\frac{u}{2}\right)^2-T_0-T_1u-T_2u^2-T_3u^3\right\}\log u\,\mathrm{d}u\\
&\quad+\int_r^\infty\frac{\mathrm{d}^4}{\mathrm{d}u^4}\left\{h\left(\frac{u}{2}\right)^2\right\}\log u\,\mathrm{d}u,
\end{aligned}$$

where

$$\begin{aligned}
&T_0=h(0)^2=1,\qquad T_1=\frac{\mathrm{d}}{\mathrm{d}u}h\left(\frac{u}{2}\right)^2_{|u=0}=0,\\
&T_2=\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left\{h\left(\frac{u}{2}\right)^2\right\}_{|u=0}=-\frac{1}{12},\qquad T_3=\frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}u^3}h\left(\frac{u}{2}\right)^2_{|u=0}=0.
\end{aligned}$$

Then

$$I=\frac{2}{r^3}-\frac{1}{2r}-3!\left\{\int_0^r\left(h\left(\frac{u}{2}\right)^2-1+\frac{1}{12}u^2\right)u^{-4}\,\mathrm{d}u+\int_r^\infty h\left(\frac{u}{2}\right)^2u^{-4}\,\mathrm{d}u\right\}.$$

Next we prove that $h(u/2)^2-1+(1/12)u^2$ takes positive values on the interval [0, 2]. For this purpose, recall that $h(u)=u/\sinh u$ is expanded for $u\in(-\pi,\pi)$ as

$$h(u)=\sum_0^\infty(-1)^k\beta_{2k}u^{2k}$$

with the coefficients

$$\beta_{2k}=\frac{2(2^{2k}-2)}{(2\pi)^{2k}}\zeta(2k).$$

Then

$$\frac{\beta_{2k+2}}{\beta_{2k}}<\frac{4-(2/2^{2n})}{(2\pi)^2(1-(2/2^{2n}))}\frac{\zeta(2k+2)}{\zeta(2k)}<\frac{3}{(2\pi)^2}<\frac{1}{12}.$$

and

$$h(x)^2=\sum(-1)^k\gamma_{2k}x^{2k},$$

with positive coefficients γ_{2k} such that

$$\gamma_{2k}=\sum_{l=0}^k\beta_{2k-2l}\beta_{2l}.$$

Hence

$$\gamma_{2k}-\gamma_{2k+2}=\beta_0(\beta_{2n}-2\beta_{2n+2})+\sum_{l=1}^k\beta_{2l}(\beta_{2k-2l}-\beta_{2k+2-2l})$$

so $\gamma_{2k}>\gamma_{2k+2}$ always.

From these estimates we see that the function $h(u/2)^2-1+(1/12)u^2$ takes positive values on the interval [0, 2], since $h(u/2)^2-1+(1/12)u^2=\sum_{k=2}^\infty(-1)^k\gamma_{2k}(u/2)^{2k}$ is the form of the alternative sum consisting of decreasing positive sequences when $u\in[0,2]$. Now we take $r=2$, then we see that $I<0$, hence we have the desired result. □

4. Heat kernel asymptotics

As an application of the formula (3.1) we calculate the heat kernel asymptotics for the three-dimensional Heisenberg manifolds, which are given in terms of Bernoulli numbers.

We know that the heat kernel $k_{\mathbf{M}}(t;x,y)$ has the asymptotic expansion:

$$k_{\mathbf{M}}(t;x,x)\sim\frac{1}{(4\pi t)^{n/2}}\{c_0(x)+c_1(x)t+c_2(x)t^2+\cdots\},\quad t\downarrow 0,$$

n is the dimension of the manifold $\mathbf{M}$.

Of course the coefficients are given in terms of quantities coming from metric tensors, but here we calculate the values

$$\mathbf{c}_k = \int_{\mathbf{M}} c_k(x)\, \mathrm{d}x$$

explicitly by the method of analytic continuation by making use of the formula (3.1), since we know that the Mellin transform of the heat kernel has poles of order 1 (at most) at the points $n/2 - k$, $k = 0, 1, 2, \ldots$, and the residue at the pole $n/2 - k$ is given by the integral $\int_{\mathbf{M}} c_k(x)\, \mathrm{d}x$.

In our cases the trace of the heat kernel $Z_{H_3/\Gamma_\ell}(t) = \int k_{H_3/\Gamma_\ell}(t; [g], [g])\, \mathrm{d}g$ is expanded as

$$\int_{H_3/\Gamma_\ell} k_{H_3/\Gamma_\ell}(t; [g], [g])\, \mathrm{d}g = Z_V(t) + Z_{T^2}(t) \sim \frac{1}{(4\pi t)^{3/2}}(\mathbf{c}_0 + \mathbf{c}_1 t + \mathbf{c}_2 t^2 + \cdots),$$

and the second term $Z_{T^2}(t)$ is that corresponding to the two-dimensional flat torus. So that it is enough to consider the Mellin transform of the first term $Z_V(t)$:

$$\begin{aligned}
&\frac{1}{\Gamma(s)} \int_0^\infty Z_V(t) t^{s-1}\, \mathrm{d}t \\
&\quad = 4\ell \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{n}{(4\pi\ell)^{2s}(n^2 + (n/4\pi\ell)(2m+1))^s} \\
&\quad = \frac{4\ell}{\Gamma(s)\Gamma(s-1)} \frac{1}{(4\pi\ell)^{2s}} \int_0^1 \left(\int_0^\infty G(u, v) u^{2s-4}\, \mathrm{d}u \right) \frac{v^{s-2}}{1+v}\, \mathrm{d}v \\
&\quad = \frac{4\ell}{\Gamma(s)\Gamma(s-1)} \frac{1}{(4\pi\ell)^{2s}} \left(\int_0^1 \left(\int_0^1 G(u, v) u^{2s-4}\, \mathrm{d}u \right) \frac{v^{s-2}}{1+v}\, \mathrm{d}v \right. \\
&\qquad \left. + \int_0^1 \left(\int_1^\infty G(u, v) u^{2s-4}\, \mathrm{d}u \right) \frac{v^{s-2}}{1+v}\, \mathrm{d}v \right).
\end{aligned} \tag{4.1}$$

Here,

$$G(u, v) = \frac{u(v+1)}{\mathrm{e}^{u(v+1)} - 1} \left\{ \frac{u}{\mathrm{e}^{u/4\pi\ell} - \mathrm{e}^{-(u/4\pi\ell)}} + \frac{uv}{\mathrm{e}^{uv/4\pi\ell} - \mathrm{e}^{-(uv/4\pi\ell)}} \right\}.$$

Since the function defined by the integral

$$\int_1^\infty G(u, v) u^{2s-4}\, \mathrm{d}u$$

is holomorphic for any $s \in \mathbb{C}$ and so the second term in expression (4.1)

$$\frac{4\ell}{\Gamma(s)\Gamma(s-1)} \frac{1}{(4\pi\ell)^{2s}} \int_0^1 \left(\int_1^\infty G(u, v) u^{2s-4}\, \mathrm{d}u \right) \frac{v^{s-2}}{1+v}\, \mathrm{d}v$$

has poles of order at most one at the points $s = 1, 0, -1, -2, \ldots$. Hence residues at these points must vanish. Consequently, it is enough to consider the first term,

$$\frac{4\ell}{\Gamma(s)\Gamma(s-1)}\frac{1}{(4\pi\ell)^{2s}}\int_0^1\left(\int_0^1 G(u,v)u^{2s-4}\,\mathrm{d}u\right)\frac{v^{s-2}}{1+v}\,\mathrm{d}v \tag{4.2}$$

for calculating the residues at the poles $s=3/2-k$, $k=0,1,2,3\ldots$. Then we have

$$\begin{aligned}&\int_0^1\left(\int_0^1 G(u,v)u^{2s-4}\,\mathrm{d}u\right)\frac{v^{s-2}}{1+v}\,\mathrm{d}v\\&\quad=4\pi\ell\int_0^1\left(\int_0^1\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\alpha_i\beta_{2j}(u(1+v))^i\left(\left(\frac{u}{4\pi\ell}\right)^{2j}+\left(\frac{uv}{4\pi\ell}\right)^{2j}\right)u^{2s-4}\,\mathrm{d}u\right)\\&\qquad\times\frac{v^{s-2}}{1+v}\,\mathrm{d}v=4\pi\ell\sum_{m=0}^{\infty}\int_0^1\frac{1}{2s-3+m}\\&\qquad\times\sum_{i+2j=m}(-1)^j\alpha_i\beta_{2j}\frac{1}{(4\pi\ell)^{2j}}(1+v)^i(1+v^{2j})\frac{v^{s-2}}{1+v}\,\mathrm{d}v.\end{aligned}\tag{4.3}$$

To calculate the residues at the poles $s=3/2-k$, $k=0,1,2,\ldots$, again it is enough to consider the terms corresponding to $m=\text{even}=2k$ in the sum (4.3).

Let $W_{i,j}(s)=\int_0^1(1+v)^{2i}(1+v^{2j})(v^{s-2}/(1+v))\,\mathrm{d}v$, then of course $W_{i,j}(s)$ is meromorphically continued to the whole complex plane and we have the following lemma.

Lemma 4.1. *$W_{i,j}(s)$ is holomorphic at $s=3/2-(i+j)$ and for $i>0$, $W_{i,j}(3/2-(i+j))=0$.*

Proof. Let $i>0$, then

$$\int_0^1(1+v)^{2i}(1+v^{2j})\frac{v^{s-2}}{1+v}\,\mathrm{d}v=\sum_{r=0}^{2i-1}{}_{2i-1}C_r\left(\frac{1}{r+s-1}+\frac{1}{r+2j+s-1}\right).$$

So at the point $s=3/2-(i+j)$ it takes the form

$$\begin{aligned}&\sum_{r=0}^{2i-1}{}_{2i-1}C_r\frac{1}{r+1/2-(i+j)}\\&\quad+\sum_{r=0}^{2i-1}{}_{2i-1}C_{2i-1-r}\frac{1}{2i-1-r+2j+3/2-(i+j)-1}.\end{aligned}$$

Hence we have $W_{i,j}(3/2-(i+j))=0$. □

Now by Lemma 4.1, to calculate the residues of the function (3.6) at the points $s=3/2-k$, it is enough to consider the function

$$4\pi\ell\sum_{k=0}^{\infty}\alpha_0\beta_{2k}\frac{1}{(4\pi\ell)^{2k}}\frac{1}{2s-3+2k}\int_0^1\frac{1+v^{2k}}{1+v}v^{s-2}\,\mathrm{d}v. \tag{4.4}$$

Let $W_k(s) = \int_0^1 [(1+v^{2k})/(1+v)]v^{s-2}\,\mathrm{d}v$, then the value at $s = 3/2 - k$,

$$W_k(\tfrac{3}{2} - k) = \mathbf{W}_k \tag{4.5}$$

is given one by one by the following lemma.

Lemma 4.2.

$$\begin{aligned} W_k\left(\frac{3}{2} - k\right) &= \int_0^1 \frac{1+v^{2k}}{1+v} v^{s-2}\,\mathrm{d}v_{|s=3/2-k} = \int_0^1 \frac{v^{s-2}}{1+v}\,\mathrm{d}v_{|s=3/2-k} + \int_0^1 \frac{v^{k-1/2}}{1+v}\,\mathrm{d}v \\ &= \sum_{r=0}^{k-1} \frac{(-1)^r}{r-k+1/2} + (-1)^k \int_0^1 \frac{1}{(1+v)\sqrt{v}}\,\mathrm{d}v + \int_0^1 \frac{v^k}{(1+v)\sqrt{v}}\,\mathrm{d}v \\ &= \sum_{r=0}^{k-1} \frac{(-1)^r}{r-k+1/2} + (-1)^k \frac{\pi}{2} + 2\sum_{r=0}^{k-1} {}_kC_r \left(\frac{k-r}{k(2r+1)} - J_r\right), \end{aligned}$$

where we put

$$J_r = \int_0^1 \frac{\theta^{2r}}{1+\theta^2}\,\mathrm{d}\theta,$$

J_r is determined by the formula

$$J_r = \sum_{i=0}^{r-1} {}_rC_i \left(\frac{r-i}{r(2i+1)} - J_i\right), \quad J_0 = \frac{\pi}{4}.$$

Also this value $\mathbf{W}_k = W_k(3/2 - k)$ is given by the formula

$$\sum_{r=0}^{k-1} \frac{(-1)^r}{r-k+1/2} + (-1)^k \frac{\pi}{2} + \sum_{j=0}^{\infty} \frac{(-1)^j}{j+k+1/2}.$$

Finally, we have the following proposition.

Proposition 4.3. *The residue* $\mathbf{c}_k = \int_{H_3/\Gamma_\ell} c_k(g)\,\mathrm{d}g$ *of the spectral zeta-function* $\zeta_{H_3/\Gamma_\ell}(s)$ *at the point* $s = 3/2 - k$, $k - 0, 1, 2, \ldots$, *is equal to*

$$\frac{4\pi}{\Gamma(3/2-k)\Gamma(1/2-k)} \frac{\beta_{2k}}{2 \cdot (4\pi\ell)^2} W_k\left(\frac{3}{2} - k\right).$$

Note that

$$\beta_{2k} = \frac{2^{2k} - 2}{(2k)!} B_{2k},$$

where $B_{2k} = (2(2k)!/(2\pi)^{2k})\zeta(2k)$ is the Bernoulli number.

5. Five-dimensional Heisenberg manifolds

So far we only considered three-dimensional cases and illustrated the procedure to calculate the determinants and heat kernel asymptotics somehow precisely. These indicate that our method for calculating the determinants and heat kernel asymptotics for higher dimensional Heisenberg manifolds would also be valid. Even though the calculations and the results are so complicated, here we state the results for the cases of five-dimensional Heisenberg manifolds.

Let $\mathfrak{h}_5$ be the five-dimensional Heisenberg Lie algebra:

$$\mathfrak{h}_5 = \mathfrak{g}_+ \oplus \mathfrak{g}_- \oplus \mathfrak{z} = \left\{ \begin{pmatrix} 0 & x_1 & x_2 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, x_i, y_i, z \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\},$$

where $\mathfrak{z}$ is the center and $[\mathfrak{g}_+, \mathfrak{g}_-] = \mathfrak{z}$.

The corresponding Lie group H_5 is realized as

$$H_5 = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, x_i, y_i, z \in \mathbb{R} \right\}. \tag{5.1}$$

As in the three-dimensional cases we only consider a left invariant Riemannian metric defined from such an inner product on the Lie algebra $\mathfrak{h}_5$ that

$$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$Z_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are orthonormal basis of $\mathfrak{h}_5$.

Also as in the three-dimensional cases let us take uniform discrete subgroups Γ_ℓ ($\ell \in \mathbb{N}$) of the form

$$\Gamma_\ell = \left\{ \begin{pmatrix} 1 & m_1 & m_2 & \dfrac{k}{2\ell} \\ 0 & 1 & 0 & n_1 \\ 0 & 0 & 1 & n_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, n_i, m_i, k \in \mathbb{Z} \right\}. \tag{5.2}$$

We identify H_5 and $\mathfrak{h}_5$ by the exponential map

$$\exp : \mathfrak{h}_5 \to H_5,$$

$$\exp : g = \begin{pmatrix} 0 & x_1 & x_2 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = x_1X_1 + x_2X_2 + y_1Y_1 + y_2Y_2 + zZ_1$$
$$\mapsto \begin{pmatrix} 1 & x_1 & x_2 & z + \dfrac{1}{2}\sum x_i y_i \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the uniform discrete subgroup $\Gamma_\ell \subset H_5$ is identified with a direct sum of two lattices $\Gamma_B \subset \mathfrak{g}_+ \oplus \mathfrak{g}_-$ and $\Gamma_V(\ell) \subset \mathfrak{z}$:

$$\Gamma_B = \left\{ \begin{pmatrix} 0 & m_1 & m_2 & 0 \\ 0 & 0 & 0 & n_1 \\ 0 & 0 & 0 & n_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, m_i, n_i \in \mathbb{Z} \right\},$$

$$\Gamma_V(\ell) = \left\{ \begin{pmatrix} 0 & 0 & 0 & \dfrac{k}{2\ell} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, k \in \mathbb{Z} \right\}, \qquad \exp(\Gamma_V(\ell) + \Gamma_B) = \Gamma_\ell.$$

The heat kernel on the whole group H_5 is given as the following form:

$$\begin{aligned} K_t(g,\tilde{g}) &= K_t(x,y,z;\tilde{x},\tilde{y},\tilde{z}) \\ &= (2\pi)^{-3}\int_{-\infty}^{+\infty} \mathrm{e}^{\sqrt{-1}\eta\{\tilde{z}-z+(1/2)([\tilde{x},y]-[x,\tilde{y}])\}}\,\mathrm{e}^{-t|\eta|^2}\left(\frac{|\eta|}{2\sinh t|\eta|}\right)^2 \\ &\quad\times\prod_{i=1}^{2}\mathrm{e}^{-(|\eta|/4)(\cosh t|\eta|/\sinh t|\eta|)\{(x_i-\tilde{x}_i)^2+(y_i-\tilde{y}_i)^2\}}\,\mathrm{d}\eta, \end{aligned} \tag{5.3}$$

where we regarded $\eta\in\mathfrak{z}^*=\mathbb{R}Z_1^*\cong\mathbb{R}$ and $g=(x,y,z)=x_1X_1+x_2X_2+yY_1+y_2Y_2+zZ_1$, and similar to $\tilde{g}$ through the exponential map.

Then the heat kernel $k_{H_5/\Gamma_\ell}(t;[g],[\tilde{g}])$ on the Heisenberg manifold $\mathbf{M}_\ell = H_5/\Gamma_\ell$, is expressed as

$$k_{H_5/\Gamma_\ell}(t;[g],[\tilde{g}]) = \sum_{\gamma\in\Gamma_\ell} K_t(\gamma\cdot g,\tilde{g}), \tag{5.4}$$

and its trace is calculated in the following form.

Theorem 5.1 (Furutani [8]).

$$\begin{aligned} Z_\ell(t) &= \int_{H_5/\Gamma_\ell} k_{H_5/\Gamma_\ell}(t;[g],[g])\,\mathrm{d}g = \sum_{\gamma\in\Gamma_\ell}\int_{F_\ell} K_t(\gamma\cdot g,g)\,\mathrm{d}g \\ &= \sum_{\mu\in\Gamma_V(\ell)^*,\mu\neq 0}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\mathrm{Vol}\left(\frac{\mathfrak{g}_+\oplus\mathfrak{g}_-}{\Gamma_B}\right)\|\mu\|^2\,\mathrm{e}^{-t\{4\pi^2\|\mu\|^2+2\pi\sum_{i=1}^{2}(2m_i+1)\|\mu\|\}} \\ &\quad+\sum_{\nu\in\Gamma_B^*}\mathrm{e}^{-4\pi^2 t\|\nu\|^2} = Z_V(t)+Z_{T^4}(t). \end{aligned}$$

Here F_ℓ denotes a fundamental domain of the uniform discrete subgroup Γ_ℓ and Γ_B^* and $\Gamma_V(\ell)^*$ are dual lattices, as before.

The second term $Z_{T^4}(t)$ is that corresponding to four-dimensional flat torus $(\mathfrak{g}_+\oplus\mathfrak{g}_-)/\Gamma_B$.

Let $\zeta_{\mathbf{M}_\ell}(s)$ be the function

$$\begin{aligned} \zeta_{\mathbf{M}_\ell}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty (Z(t)-1)t^{s-1}\,\mathrm{d}t = \frac{1}{\Gamma(s)}\int_0^\infty Z_V(t)t^{s-1}\,\mathrm{d}t \\ &\quad+\frac{1}{\Gamma(s)}\int_0^\infty (Z_{T^4}(t)-1)t^{s-1}\,\mathrm{d}t = \zeta_{V,\ell}(s)+\zeta_{T^4}(s), \end{aligned}$$

then we have the following theorem.

Theorem 5.2.

$$\mathrm{Det}\,\Delta_{H_5/\Gamma_\ell} = \mathrm{Det}\,\Delta_{T^4}\,\mathrm{e}^{-\zeta'_{V,\ell}(0)},$$

where $\zeta'_{V,\ell}(0)$ *is given in* Proposition 5.4.

Corresponding to the formula (3.1) we have an integral representation of the function $\zeta_{V,\ell}(s)$:

$$\begin{aligned}\zeta_{V,\ell}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty Z_V(t)t^{s-1}\,\mathrm{d}t\\ &= 8\ell^2\sum_{n=1}^\infty\sum_{m_1=0}^\infty\sum_{m_2=0}^\infty \frac{n^2}{(4\pi\ell)^{2s}(n^2+(n/4\pi\ell)(2m_1+1+2m_2+1))^s}\\ &= \frac{8\ell^2}{\Gamma(s)\Gamma(s-2)}\frac{1}{(4\pi\ell)^{2s}}\int_0^\infty\int_0^\infty \frac{x+y}{\mathrm{e}^{x+y}-1}\left(\frac{x}{\mathrm{e}^{x/4\pi\ell}-\mathrm{e}^{-(x/4\pi\ell)}}\right)^2\\ &\quad\times\frac{(xy)^{s-3}}{x+y}\,\mathrm{d}x\,\mathrm{d}y = \frac{8\ell^2s^2(s-1)(s-2)}{\Gamma(s+1)^2(4\pi\ell)^{2s}}\,\frac{4\pi^2\ell^2}{(2s-5)(2s-4)(2s-3)\times(2s-2)(2s-1)2s}\\ &\quad\times\int_0^1\int_0^\infty \frac{\mathrm{d}^6}{\mathrm{d}u^6}\left\{g(u(1+v))\left(h\left(\frac{u}{4\pi\ell}\right)^2+h\left(\frac{uv}{4\pi\ell}\right)^2\right)\right\}u^{2s}\frac{v^{s-3}}{1+v}\,\mathrm{d}u\,\mathrm{d}v.\end{aligned} \tag{5.5}$$

Proposition 5.3.

$$\begin{aligned}&\int_0^1\int_0^\infty \frac{\mathrm{d}^6}{\mathrm{d}u^6}\left\{g(u(1+v))\left(h\left(\frac{u}{4\pi\ell}\right)^2+h\left(\frac{uv}{4\pi\ell}\right)^2\right)\right\}u^{2s}\frac{v^{s-3}}{1+v}\,\mathrm{d}u\,\mathrm{d}v\\ &\quad= 2\int_0^\infty \frac{\mathrm{d}^6}{\mathrm{d}u^6}\left\{h\left(\frac{u}{2}\right)^3\left(\cosh\frac{u}{2}\right)\left(h\left(\frac{u}{4\pi\ell}\right)^2+1\right)\right.\\ &\qquad\left.-\frac{1}{3}\left(\frac{u}{4\pi\ell}\right)^2\mathrm{e}^{-u/2}h\left(\frac{u}{2}\right)\right\}\log u\,\mathrm{d}u = 2\int_0^\infty\frac{\mathrm{d}^6}{\mathrm{d}u^6}\left\{\left(\frac{u/2}{\sinh u/2}\right)^3\left(\cosh\frac{u}{2}\right)\right.\\ &\qquad\left.\times\left(\left(\frac{u/4\pi\ell}{\sinh u/4\pi\ell}\right)^2+1\right)-\frac{1}{3}\left(\frac{u}{4\pi\ell}\right)^2\mathrm{e}^{-u/2}\frac{u/2}{\sinh u/2}\right\}\log u\,\mathrm{d}u+0(s).\end{aligned}$$

Proposition 5.4.

$$\begin{aligned}\zeta'_{V,\ell}(0) &= -\frac{8}{15}\pi^2\ell^4\int_0^\infty\frac{\mathrm{d}^6}{\mathrm{d}u^6}\left\{\left(\frac{u/2}{\sinh u/2}\right)^3\cosh\frac{u}{2}\left(\left(\frac{u/4\pi\ell}{\sinh u/4\pi\ell}\right)^2+1\right)\right.\\ &\qquad\left.-\frac{1}{3}\left(\frac{u}{4\pi\ell}\right)^2\mathrm{e}^{-u/2}\frac{u/2}{\sinh u/2}\right\}\log u\,\mathrm{d}u.\end{aligned}$$

Corollary 5.5.

$$\lim_{\ell\to\infty}\mathrm{Det}\,\Delta_{H_5/\Gamma_\ell} = \lim_{\ell\to\infty}\mathrm{Det}\,\Delta_{T^4}\,\mathrm{e}^{-\zeta'_{V,\ell}(0)} = 0.$$

Proof. Put

$$I(\ell) = \int_0^\infty \frac{\mathrm{d}^6}{\mathrm{d}u^6}\left\{\left(\frac{u/2}{\sinh u/2}\right)^3 \cosh\frac{u}{2}\left(\left(\frac{u/4\pi\ell}{\sinh u/4\pi\ell}\right)^2 + 1\right)\right.$$
$$\left. - \frac{1}{3}\left(\frac{u}{4\pi\ell}\right)^2 \mathrm{e}^{-u/2}\frac{u/2}{\sinh u/2}\right\} \log u\,\mathrm{d}u.$$

Since

$$\lim_{\ell\to\infty} I(\ell) = 2\int_0^\infty \frac{\mathrm{d}^6}{\mathrm{d}u^6}\left\{\left(\frac{u/2}{\sinh u/2}\right)^3 \cosh\frac{u}{2}\right\} \log u\,\mathrm{d}u, \tag{5.6}$$

it is enough to see the sign of this integral as we did before in Corollary 3.9 to determine the behavior of the determinant $\mathrm{Det}\,\Delta_{H_5/\Gamma_\ell}$ when $\ell \to \infty$. Then, by a similar calculation in Corollary 3.9 we have an expression of the integral

$$\int_0^\infty \frac{\mathrm{d}^6}{\mathrm{d}u^6}\left\{\left(\frac{u/2}{\sinh u/2}\right)^3 \cosh\frac{u}{2}\right\} \log u\,\mathrm{d}u$$
$$= \frac{4!}{r^5} - \frac{1}{2r} - 5!\int_0^r \left(\left(\frac{u/2}{\sinh u/2}\right)^3 \cosh\frac{u}{2} - 1 + \frac{1}{2^4\cdot 3\cdot 5}u^4\right) u^{-6}\,\mathrm{d}u$$
$$- 5!\int_r^\infty \left(\left(\frac{u/2}{\sinh u/2}\right)^3 \cosh\frac{u}{2}\right) u^{-6}\,\mathrm{d}u.$$

So putting $r = 2\cdot\sqrt[4]{3}$, we have

$$\int_0^\infty \frac{\mathrm{d}^6}{\mathrm{d}u^6}\left\{\left(\frac{u/2}{\sinh u/2}\right)^3 \cosh\frac{u}{2}\right\} \log u\,\mathrm{d}u$$
$$= -5!\int_0^{2\sqrt[4]{3}} \left(\left(\frac{u/2}{\sinh u/2}\right)^3 \cosh\frac{u}{2} - 1 + \frac{1}{2^4\cdot 3\cdot 5}u^4\right) u^{-6}\,\mathrm{d}u$$
$$- 5!\int_{2\sqrt[4]{3}}^\infty \left(\left(\frac{u/2}{\sinh u/2}\right)^3 \cosh\frac{u}{2}\right) u^{-6}\,\mathrm{d}u.$$

It is clear that the second integrand takes always positive values on the positive real axis. We can also prove that the integrand in the first integral takes positive values on the positive real axis. Since the coefficients of the Taylor expansion of the function

$$\cosh x - \left(1 - \frac{1}{15}x^4\right)\left(\frac{\sinh x}{x}\right)^3 = \sum_{n=3}^\infty a_n x^{2n} \tag{5.7}$$

are given as

$$a_n = \frac{1}{(2n)!} + \frac{3^{2n-2}-1}{20\cdot(2n-1)!} - \frac{3}{4}\frac{3^{2n+2}-1}{(2n+3)!} \tag{5.8}$$

and all take positive values, which we can see from the expression of a_n for $n \geq 4$,

$$a_n = \frac{1}{20 \cdot (2n+3)!}\{(3^{2n-2} - 1)((2n+3)(2n+2)(2n+1)(2n) - 15 \cdot 3^4) + 20 \cdot (2n+3)(2n+2)(2n+1) - 15 \cdot (3^4 - 1)\},$$

$$a_4 = \frac{1}{20 \cdot (11)!}\{(3^6 - 1)(11 \cdot 10 \cdot 9 \cdot 8 - 15 \cdot 3^4) + 20 \cdot 11 \cdot 10 \cdot 9 - 15 \cdot (3^4 - 1)\}$$

and for $n = 3$, $a_3 = 4/189$. From these facts we can prove the desired result. □

Finally, we list the heat asymptotics for five-dimensional Heisenberg manifolds $\mathbf{M}_\ell$. Let $\mathbf{c}_k$ be the coefficients of the asymptotic expansion of the heat kernel $k_{H_5/\Gamma_\ell}(t; x, y)$ of the five-dimensional Heisenberg manifolds $\mathbf{M}_\ell = H_5/\Gamma_\ell$:

$$\int_{\mathbf{M}_\ell} k_{\mathbf{M}_\ell}(t; x, x)\, \mathrm{d}x \sim \frac{1}{(4\pi t)^{5/2}}\{\mathbf{c}_0 + \mathbf{c}_1 t + \cdots + \mathbf{c}_k t^k + \cdots\}.$$

Let us denote the Taylor expansion of the function $h(x)^2$ as

$$h(x)^2 = \left(\frac{x}{\sinh x}\right)^2 = \sum_{k=0}^{\infty}(-1)^k \delta_k x^{2k}, \tag{5.9}$$

then δ_k is given by

$$\delta_k = \sum_{j=0}^{k} \beta_{2j}\beta_{2(k-j)} = \sum_{j=0}^{k} \frac{4(2^{2(k-j)} - 2)(2^{2j} - 2)}{(2\pi)^k}\zeta(2(k-j))\zeta(2j). \tag{5.10}$$

Proposition 5.6.

$$\mathbf{c}_k = \frac{(2k-5)!!(2k-1)!!}{2^{2k+4}}\frac{(-1)^k}{\pi^3 \ell}\delta_k \mathbf{W}_k, \tag{5.11}$$

where $\mathbf{W}_k$ *is given in* (4.5).

6. A formula for product manifolds

Let $(\mathbf{M}, g)$ and $(\mathbf{N}, h)$ be closed Riemannian manifolds, then the Laplacian $\Delta_{\mathbf{M}\times\mathbf{N}}$ on the product Riemannian manifold $\mathbf{M} \times \mathbf{N}$ is of the form $\Delta_{\mathbf{M}} \otimes \mathrm{Id} + \mathrm{Id} \otimes \Delta_{\mathbf{N}}$ and the spectrum $\mathrm{Spec}(\Delta_{\mathbf{M}\times\mathbf{N}})$ is given by

$$\mathrm{Spec}(\Delta_{\mathbf{M}\times\mathbf{N}}) = \{\lambda_m + \mu_n | 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \in \mathrm{Spec}(\Delta_{\mathbf{M}}),\ \text{and}\ 0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \in \mathrm{Spec}(\Delta_{\mathbf{N}})\}.$$

In this section we give a formula of the zeta-regularized determinant of the Laplacian on the product Riemannian manifold $\mathbf{M} \times \mathbf{N}$ in terms of each value of the zeta-regularized determinant and heat invariants.

The spectral zeta-function $\zeta_{\mathbf{M}\times\mathbf{N}}(s)$ for the product Riemannian manifold $\mathbf{M}\times\mathbf{N}$ is given by

$$\zeta_{\mathbf{M}\times\mathbf{N}}(s) = \sum_{m,n=0,(m,n)\neq 0}^{\infty} \frac{1}{(\lambda_m + \mu_n)^s}.$$

We express this as

$$\begin{aligned}
&\sum_{m,n=0,(m,n)\neq 0}^{\infty} \frac{1}{(\lambda_m + \mu_n)^s} \\
&\quad = \frac{1}{\Gamma(s)} \sum_{m=1}^{\infty} \frac{1}{\lambda_m^s} \sum_{n=0}^{\infty} \int_0^{\infty} \mathrm{e}^{-(1+(\mu_n/\lambda_m))t} t^{s-1}\,\mathrm{d}t + \zeta_{\mathbf{N}}(s) \\
&\quad = \frac{1}{\Gamma(s)} \sum_{m=1}^{\infty} \frac{1}{\lambda_m^s} \int_0^{\infty} \left\{ \sum_{n=0}^{\infty} \mathrm{e}^{-(t/\lambda_m)\cdot\mu_n} - \left(\frac{\lambda_m}{4\pi t}\right)^{N/2} \sum_{i=0}^{(N+M)/2} \mathbf{b}_i \cdot \left(\frac{t}{\lambda_m}\right)^i \right\} \\
&\qquad \times \mathrm{e}^{-t} t^{s-1}\,\mathrm{d}t + \frac{1}{\Gamma(s)} \sum_{m=1}^{\infty} \frac{1}{\lambda_m^s} \int_0^{\infty} \left(\frac{\lambda_m}{4\pi t}\right)^{N/2} \\
&\qquad \times \sum_{i=0}^{(N+M)/2} \mathbf{b}_i \cdot \left(\frac{t}{\lambda_m}\right)^i \mathrm{e}^{-t} t^{s-1}\,\mathrm{d}t + \zeta_{\mathbf{N}}(s)
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
&\sum_{m,n=0,(m,n)\neq 0}^{\infty} \frac{1}{(\lambda_m + \mu_n)^s} \\
&\quad = \mathcal{Q}_0(s) + \sum_{i=0}^{(N+M)/2} \frac{\mathbf{b}_i \cdot \Gamma(s+i-N/2)}{(4\pi)^{N/2}\cdot\Gamma(s)} \cdot \zeta_{\mathbf{M}}\left(s+i-\frac{N}{2}\right) + \zeta_{\mathbf{N}}(s),
\end{aligned} \tag{6.2}$$

where $N=\dim\mathbf{N}$, $M=\dim\mathbf{M}$ and $\{\mathbf{b}_i\}_{i=0}^{\infty}$ are the coefficients of asymptotic expansion of the trace of the heat kernel $k_{\mathbf{N}}(t;x,y)$ on $\mathbf{N}$:

$$k_{\mathbf{N}}(t) = \int_{\mathbf{N}} k_{\mathbf{N}}(t;x,x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \mathrm{e}^{-t\mu_n} \sim \left(\frac{1}{4\pi t}\right)^{N/2} \{\mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \mathbf{b}_3 t^3 + \cdots\}. \tag{6.3}$$

We also denote by $\{\mathbf{a}_i\}_{i=0}^{\infty}$ the coefficients of the asymptotic expansion of trace of the heat kernel $k_{\mathbf{M}}(t;x,y)$ on the manifold $\mathbf{M}$:

$$\begin{aligned}
k_{\mathbf{M}}(t) &= \int_{\mathbf{M}} k_{\mathbf{M}}(t;x,x)\,\mathrm{d}x \\
&= \sum_{m=0}^{\infty} \mathrm{e}^{-t\lambda_m} \sim \left(\frac{1}{4\pi t}\right)^{M/2} \{\mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3 + \cdots\}.
\end{aligned} \tag{6.4}$$

Since

$$\left|\left\{k_{\mathbf{N}}\left(\frac{t}{\lambda_m}\right)-\left(\frac{\lambda_m}{4\pi t}\right)^{N/2}\sum_{i=0}^{(N+M)/2}\mathbf{b}_i\cdot\left(\frac{t}{\lambda_m}\right)^{i}\right\}\right| = \mathrm{O}\left(\left(\frac{t}{\lambda_m}\right)^{[(N+M)/2]+1-N/2}\right),$$

the first term $\mathcal{Q}_0(s)$ is holomorphic on the domain $\{s\in\mathbb{C}|\mathfrak{Re}(s)>-1/2\}$, at least under this expression for any case of the dimensions of the two manifolds. So we can put $s=0$ in (6.1) and we have

$$\mathcal{Q}_0(0)=0,$$
$$\mathcal{Q}_0'(0)=\sum_{m=1}^{\infty}\int_0^{\infty}\left\{k_{\mathbf{N}}\left(\frac{t}{\lambda_m}\right)-\left(\frac{\lambda_m}{4\pi t}\right)^{N/2}\sum_{i=0}^{(N+M)/2}\mathbf{b}_i\cdot\left(\frac{t}{\lambda_m}\right)^{i}\right\}\mathrm{e}^{-t}t^{-1}\,\mathrm{d}t. \tag{6.5}$$

Next we put the second term as

$$\mathcal{Q}_1(s)=\sum_{i=0}^{(N+M)/2}\frac{\mathbf{b}_i\cdot\Gamma(s+i-N/2)}{(4\pi)^{N/2}\cdot\Gamma(s)}\cdot\zeta_{\mathbf{M}}\left(s+i-\frac{N}{2}\right)=\sum_{i=0}^{(N+M)/2}\mathfrak{q}_i(s) \tag{6.6}$$

$$\mathcal{Q}_1(s)=\sum_{i=0}^{(N+M)/2}\frac{\mathbf{b}_i}{(4\pi)^{N/2}\cdot\Gamma(s)}\int_0^{\infty}(k_{\mathbf{M}}(t)-1)t^{s+i-N/2-1}\,\mathrm{d}t. \tag{6.7}$$

Remark 6.1. By the asymptotic expansion (6.4) it is well known that the Mellin transformation $\int_0^{\infty}(k_{\mathbf{M}}(t)-1)t^{s-1}\,\mathrm{d}t$ is meromorphically continued to the whole complex plane and has possible poles of order 1 at points $M/2-i$, $i=0,1,\ldots$ with the residue

$$\frac{\mathbf{a}_i}{(4\pi)^{M/2}}$$

when $\dim\mathbf{M}$ is odd, and when $\dim\mathbf{M}$ is even then the residue at the pole $M/2-i$, $i\neq M/2$ is

$$\frac{\mathbf{a}_i}{(4\pi)^{M/2}},$$

and at $s=0$ the residue is

$$\frac{\mathbf{a}_{M/2}}{(4\pi)^{M/2}}-1.$$

By the remark above we can describe the derivative at $s=0$ of the each term

$$\mathfrak{q}_i(s)=\frac{\mathbf{b}_i\cdot\Gamma(s+i-N/2)}{(4\pi)^{N/2}\cdot\Gamma(s)}\cdot\zeta_{\mathbf{M}}\left(s+i-\frac{N}{2}\right)$$

in (6.6) in terms of the spectral zeta-function $\zeta_{\mathbf{M}}(s)$. For this purpose we consider four cases separately.

Proposition 6.2. *Let* $\dim \mathbf{M} = M$ *be even and* $\dim \mathbf{N} = N$ *odd. Since* $\zeta_{\mathbf{M}}(s)$ *is holomorphic at each point* $i - N/2$ *we have*

$$\mathcal{Q}_1'(0) = \sum_{i=0}^{(N+M)/2} \mathfrak{q}_i'(0) = \sum_{i=0}^{(N+M)/2} \frac{\mathbf{b}_i \cdot \Gamma(i - N/2)}{(4\pi)^{N/2}} \cdot \zeta_{\mathbf{M}}\left(i - \frac{N}{2}\right).$$

Proposition 6.3. *Let* $\dim \mathbf{M} = M$ *be odd and* $\dim \mathbf{N} = N$ *even. Then*:

- *for* $0 \le i < N/2$:

$$\mathfrak{q}_i'(0) = \frac{(-1)^{N/2-i}\mathbf{b}_i}{(4\pi)^{N/2}(N/2 - i)!} \cdot \zeta_{\mathbf{M}}'\left(i - \frac{N}{2}\right),$$

- *for* $i = N/2$:

$$\mathfrak{q}_{N/2}'(0) = \frac{\mathbf{b}_{N/2}}{(4\pi)^{N/2}} \cdot \zeta_{\mathbf{M}}'(0),$$

- *for* $N/2 < i \le (N + M)/2$:

$$\mathfrak{q}_i'(0) = \frac{\mathbf{b}_i (i - N/2 - 1)!}{(4\pi)^{N/2}} \cdot \zeta_{\mathbf{M}}\left(i - \frac{N}{2}\right).$$

Hence

$$\begin{aligned}\mathcal{Q}_1'(0) &= \sum_{i=0}^{N/2-1} \frac{(-1)^{N/2-i}\mathbf{b}_i}{(4\pi)^{N/2}(N/2 - i)!} \cdot \zeta_{\mathbf{M}}'(0) + \frac{\mathbf{b}_{N/2}}{(4\pi)^{N/2}} \cdot \zeta_{\mathbf{M}}'(0) \\ &\quad + \sum_{i=N/2+1}^{(N+M)/2} \frac{\mathbf{b}_i}{(4\pi)^{N/2}(i - N/2 - 1)!} \cdot \zeta_{\mathbf{M}}\left(i - \frac{N}{2}\right).\end{aligned}$$

Proposition 6.4. *Let both of* $\dim \mathbf{M} = M$ *and* $\dim \mathbf{N} = N$ *be odd. Then,*

$$\begin{aligned}\mathfrak{q}_i'(0) &= \frac{\mathbf{b}_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} + \left(\lim_{s\to 0} \Gamma\left(i - \frac{N}{2}\right) \cdot \zeta_{\mathbf{M}}\left(s + i - \frac{N}{2}\right) \right.\right. \\ &\qquad \left.\left. - \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} \frac{1}{s} \right) \right\}.\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{Q}_1'(0) &= \sum_{i=0}^{(N+M)/2} \frac{\mathbf{b}_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} \right. \\ &\qquad \left. + \left(\lim_{s\to 0} \Gamma\left(i - \frac{N}{2}\right) \cdot \zeta_{\mathbf{M}}\left(s + i - \frac{N}{2}\right) - \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} \frac{1}{s} \right) \right\}.\end{aligned}$$

Proposition 6.5. *Let* $\dim \mathbf{M} = M$ *and* $\dim \mathbf{N} = N$ *be both even. Then*:

- *for* $0 \le i < N/2$:

$$\mathfrak{q}_i'(0) = \frac{\mathbf{b}_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} + \lim_{s\to 0} \Gamma\left(s+i-\frac{N}{2}\right) \cdot \zeta_{\mathbf{M}}\left(s+i-\frac{N}{2}\right) - \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} \frac{1}{s} \right\},$$

- *for* $i = N/2$:

$$q_{N/2}'(0) = \frac{\mathbf{b}_{N/2}}{(4\pi)^{N/2}} \cdot \zeta_{\mathbf{M}}'(0),$$

- *for* $N/2 < i \le (N+M)/2$:

$$\mathfrak{q}_i'(0) = \frac{\mathbf{b}_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} + \lim_{s\to 0} \left(\frac{N}{2}-i-1\right)! \cdot \zeta_{\mathbf{M}}\left(s+i-\frac{N}{2}\right) - \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} \frac{1}{s} \right\}.$$

Hence,

$$\begin{aligned}\mathcal{Q}_1'(0) = {} & \sum_{i=0}^{N/2-1} \frac{\mathbf{b}_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} + \lim_{s\to 0} \Gamma\left(s+i-\frac{N}{2}\right) \cdot \zeta_{\mathbf{M}}\left(s+i-\frac{N}{2}\right) - \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} \frac{1}{s} \right\} \\ & + \frac{\mathbf{b}_{N/2}}{(4\pi)^{N/2}} \cdot \zeta_{\mathbf{M}}'(0) + \sum_{i=N/2+1}^{(N+M)/2} \frac{\mathbf{b}_i}{(4\pi)^{N/2}} \left\{ -\Gamma'(1) \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} \right. \\ & \left. + \lim_{s\to 0} \left(\frac{N}{2}-i-1\right)! \cdot \zeta_{\mathbf{M}}\left(s+i-\frac{N}{2}\right) - \frac{\mathbf{a}_{\{(N+M)/2-i\}}}{(4\pi)^{M/2}} \frac{1}{s} \right\}.\end{aligned}$$

Remark 6.6. $-\Gamma'(1) = -\int_0^\infty \mathrm{e}^{-t} \log t \,\mathrm{d}t = \mathbf{C_e} = 0.57721\cdots =$ Euler's constant.

We can now write down an expression of the value $\mathrm{Det}\,\Delta_{\mathbf{M}\times\mathbf{N}}$ corresponding to the each case (6.2)–(6.5). Here we only state for the case that both of $\dim \mathbf{M}$ and $\dim \mathbf{N}$ are even. In the next section we state a special case of $\mathbf{N} = S^1$.

Theorem 6.7. *Let* $\dim \mathbf{N}$ *and* $\dim \mathbf{M}$ *be both even, then*

$$\begin{aligned}\mathrm{Det}\,\Delta_{\mathbf{M}\times\mathbf{N}} &= \mathrm{e}^{-\mathcal{Q}_0'(0)}\,\mathrm{e}^{-\mathcal{Q}_1'(0)}\,\mathrm{Det}\,\Delta_{\mathbf{N}} \\ &= \mathrm{Det}\,\Delta_{\mathbf{N}} \prod_{m=1}^{\infty} \mathrm{e}^{-\int_0^\infty \{k_{\mathbf{N}}(t/\lambda_m) - (\lambda_m/4\pi t)^{N/2} \sum_{i=0}^{(N+M)/2} \mathbf{b}_i (t/\lambda_m)^i\}\,\mathrm{e}^{-t} t^{-1}\,\mathrm{d}t} \\ &\quad \times \prod_{i=0}^{N/2-1} \mathrm{e}^{-\mathbf{C_e}\cdot\mathbf{b}_i\cdot\mathbf{a}_{\{(N+M)/2-i\}}/(4\pi)^{(N+M)/2}}\end{aligned}$$

$$\times \prod_{i=0}^{N/2-1} \exp[-(\mathbf{b}_i/(4\pi)^{N/2})\{\lim_{s\to 0} \Gamma(s+i-N/2)\cdot \zeta_{\mathbf{M}}(s+i-N/2)$$
$$- (\mathbf{a}_{\{(N+M)/2-i\}}/(4\pi)^{M/2})(1/s)\}](\mathrm{Det}\, \Delta_{\mathbf{M}})^{\mathbf{b}_{N/2}/(4\pi)^{N/2})}$$
$$\times \prod_{i=N/2+1}^{(N+M)/2} \mathrm{e}^{-\mathbf{C}_{\mathbf{e}}\cdot\mathbf{b}_i\cdot\mathbf{a}_{\{(N+M)/2-i\}}/(4\pi)^{(N+M)/2}}$$
$$\times \prod_{i=N/2+1}^{(N+M)/2} \exp[-(\mathbf{b}_i/(4\pi)^{N/2})\{\lim_{s\to 0}(N/2-i-1)!\cdot \zeta_{\mathbf{M}}(s+i-N/2)$$
$$- (\mathbf{a}_{\{(N+M)/2-i\}}/(4\pi)^{M/2})(1/s)\}].$$

By interchanging $\mathbf{N}$ *and* $\mathbf{M}$ *we have another expression of* $\mathrm{Det}\, \Delta_{\mathbf{N}\times\mathbf{M}}$:

$$\mathrm{Det}\, \Delta_{\mathbf{M}\times\mathbf{N}} = \mathrm{Det}\, \Delta_{\mathbf{M}} \prod_{n=1}^{\infty} \mathrm{e}^{-\int_0^\infty \{k_{\mathbf{M}}(t/\mu_n)-(\mu_n/4\pi t)^{M/2}\sum_{i=0}^{(M+N)/2} \mathbf{a}_i(t/\mu_n)^i\}\,\mathrm{e}^{-t}t^{-1}\,\mathrm{d}t}$$
$$\times \prod_{i=0}^{M/2-1} \mathrm{e}^{-\mathbf{C}_{\mathbf{e}}\cdot\mathbf{a}_i\cdot\mathbf{b}_{\{(N+M)/2-i\}}/(4\pi)^{(M+N)/2}}$$
$$\times \prod_{i=0}^{M/2-1} \mathrm{e}^{-(\mathbf{a}_i/(4\pi)^{M/2})\{\lim_{s\to 0}\Gamma(s+i-M/2)\cdot\zeta_{\mathbf{N}}(s+i-M/2)-(\mathbf{b}_{\{(N+M)/2-i\}}/(4\pi)^{N/2})(1/s)\}}$$
$$\times (\mathrm{Det}\, \Delta_{\mathbf{N}})^{\mathbf{a}_{M/2}/(4\pi)^{M/2}} \prod_{i=M/2+1}^{(N+M)/2} \mathrm{e}^{-\mathbf{C}_{\mathbf{e}}\cdot\mathbf{a}_i\cdot\mathbf{b}_{\{(N+M)/2-i\}}/(4\pi)^{(N+M)/2}}$$
$$\times \prod_{i=M/2+1}^{(N+M)/2} \mathrm{e}^{-(\mathbf{a}_i/(4\pi)^{M/2})\{\lim_{s\to 0}(M/2-i-1)!\cdot\zeta_{\mathbf{N}}(s+i-M/2)-(\mathbf{b}_{\{(N+M)/2-i\}}/(4\pi)^{N/2})(1/s)\}}.$$

7. Product manifold M × S^1

The formula we gave in the last section says that even in the product manifold case the zeta-regularized determinant is not expressed in a simple way in terms of each zeta-regularized determinant. In the paper [7] a formula of the zeta-regularized determinant for a manifold of the type $\mathbf{M} \times S^1$ is given, in fact a formula is derived for a higher order operator of the form with the variable in S^1 being separated, by reducing the problem to a boundary value problem on the manifold $\mathbf{M} \times [0, 1]$. In this section we give a direct proof of this formula by restricting ourselves to the case of Laplacians on $\mathbf{M} \times S^1$ by following the same line as we did in the last section. Here in this special case the Poisson summation formula allows us to make an integration of the term (6.5) and we arrive at a precise formula.

Now in this case the Laplacian $\Delta_{\mathbf{M}\times S^1}$ is given by $\Delta_{\mathbf{M}\times S^1} = \Delta_{\mathbf{M}} - (\mathrm{d}^2/\mathrm{d}x^2)$, where we regard $S^1 \cong \mathbb{R}/(2\pi\ell\cdot\mathbb{Z})$, $x \in \mathbb{R}$, $\ell > 0$ and the spectrum are

$$\left\{ \lambda_n + \left(\frac{k}{\ell}\right)^2 \middle| \, 0 = \lambda_0 < \lambda_1 \le \lambda_2, \ldots, \text{ are the spectrum of } \mathbf{M}, \text{ and } k \in \mathbb{Z} \right\}.$$

Theorem 7.1.

$$\operatorname{Det} \Delta_{\mathbf{M}\times S^1} = 4\pi^2\ell^2 \mathbf{C}_{\mathbf{M}} \prod_{n=1}^{\infty} |(1-\mathrm{e}^{-2\pi\ell\sqrt{\lambda_n}})|^2, \tag{7.1}$$

where the constant $\mathbf{C}_{\mathbf{M}}$ *is given by*:

(a) *when* $\dim \mathbf{M} = even = 2m$, *then*

$$\mathbf{C}_{\mathbf{M}} = \mathrm{e}^{2\pi\ell\cdot\zeta_{\mathbf{M}}(-1/2)},$$

(b) *when* $\dim \mathbf{M} = odd = 2m+1$, *then*

$$\begin{aligned}\log \mathbf{C}_{\mathbf{M}} &= -\sqrt{\pi}\ell \left\{ \lim_{s\to 0} \left(2\sqrt{\pi}\cdot\zeta_{\mathbf{M}}\left(s-\frac{1}{2}\right) + \frac{\mathbf{a}_{(M+1)/2}}{(4\pi)^{(M+1)/2}}\frac{1}{s}\right)\right. \\ &\qquad \left. + \frac{\Gamma'(1)}{(4\pi)^{M/2}}\mathbf{a}_{(M+1)/2}\right\}.\end{aligned}$$

Here $\mathbf{a}_{(M+1)/2}$ denotes $((M+1)/2)$th coefficient of the asymptotic expansion of the heat kernel $Z_{\mathbf{M}}(t)$ on $\mathbf{M}$.

Proof. We express the spectral zeta-function $\zeta_{\mathbf{M}\times S^1}(s)$ as

$$\begin{aligned}\zeta_{\mathbf{M}\times S^1}(s) &= \sum_{n=1}^{\infty}\sum_{k\in\mathbb{Z}} \frac{1}{(\lambda_n + (k/\ell)^2)^s} + 2\ell^{2s}\cdot\zeta(2s) \\ &= \frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^s}\int_0^{\infty}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-(1+(k/\ell)^2(1/\lambda_n))x} x^{s-1}\,\mathrm{d}x + 2\ell^{2s}\cdot\zeta(2s).\end{aligned}$$

Then by using the Poisson's summation formula this equals to

$$\begin{aligned}&\frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^s}\int_0^{\infty}\sum_{k\in\mathbb{Z}}\sqrt{\frac{\pi\lambda_n\ell^2}{x}}\mathrm{e}^{-\pi^2\lambda_n k^2\ell^2/x}\,\mathrm{e}^{-x}x^{s-1}\,\mathrm{d}x + 2\ell^{2s}\cdot\zeta(2s) \\ &= \frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^s}\int_0^{\infty}\sum_{k\in\mathbb{Z},k\neq 0}\sqrt{\frac{\pi\lambda_n\ell^2}{x}}\,\mathrm{e}^{-\pi^2\lambda_n k^2\ell^2/x}\,\mathrm{e}^{-x}x^{s-1}\,\mathrm{d}x \\ &\quad + \frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^s}\int_0^{\infty}\sqrt{\frac{\pi\lambda_n\ell^2}{x}}\,\mathrm{e}^{-x}x^{s-1}\,\mathrm{d}x + 2\ell^{2s}\cdot\zeta(2s) = \mathcal{P}_0(s) \\ &\quad + \frac{\sqrt{\pi}\ell\cdot\Gamma(s-1/2)}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{s-1/2}} + 2\ell^{2s}\cdot\zeta(2s),\end{aligned}$$

where we put

$$\mathcal{P}_0(s) = \frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^s}\int_0^{\infty}\sum_{k\in\mathbb{Z},k\neq 0}\sqrt{\frac{\pi\lambda_n\ell^2}{x}}\,\mathrm{e}^{-\pi^2\lambda_n k^2\ell^2/x}\,\mathrm{e}^{-x}x^{s-1}\,\mathrm{d}x$$
$$= \frac{2\sqrt{\pi}\ell}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{s-1/2}}\int_0^{\infty}\sum_{k=1}^{\infty}\mathrm{e}^{-\pi^2\lambda_n k^2\ell^2/x}\,\mathrm{e}^{-x}x^{s-3/2}\,\mathrm{d}x.$$

We denote the second term

$$\frac{\sqrt{\pi}\ell\cdot\Gamma(s-1/2)}{\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{s-1/2}} = \frac{\sqrt{\pi}\ell\cdot\Gamma(s-1/2)}{\Gamma(s)}\cdot\zeta_{\mathbf{M}}\left(s-\frac{1}{2}\right)$$

by $\mathcal{P}_1(s)$.

Since the sum $\sum_{k=1}^{\infty}\mathrm{e}^{-\pi^2\lambda_n k^2\ell^2/x}$ in the integrand satisfies the asymptotics

$$\sum_{k=1}^{\infty}\mathrm{e}^{-\pi^2\lambda_n k^2\ell^2/x} = \mathrm{O}\left(\left(\frac{x}{\lambda_n}\right)^N\right)$$

for any $N\in\mathbb{N}$, or the coefficients $\mathbf{b}_i$ are all zero except $\mathbf{b}_0 = 2\pi\ell$, the function $\mathcal{P}_0(s)$ is holomorphic on the whole complex plane and we have

$$\mathcal{P}_0(0) = 0, \qquad \mathcal{P}_0'(0) = 2\sqrt{\pi}\ell\sum_{n=1}^{\infty}\sqrt{\lambda_n}\int_0^{\infty}\sum_{k=1}^{\infty}\mathrm{e}^{-\pi^2\lambda_n k^2\ell^2/x}\,\mathrm{e}^{-x}x^{-3/2}\,\mathrm{d}x.$$

To calculate the value $\mathcal{P}_0'(0)$ recall a formula of a *modified Bessel function* $K_{1/2}(z) = K_{-1/2}(z)$ (see [1]):

$$\int_0^{\infty}\mathrm{e}^{-(t+z^2/4t)}\frac{1}{\sqrt{t}}\,\mathrm{d}t = \sqrt{\pi}\,\mathrm{e}^{-z} = \sqrt{2z}K_{1/2}(z). \tag{7.2}$$

Now we have

$$\mathcal{P}_0'(0) = -2\sum_{n=1}^{\infty}\log(1-\mathrm{e}^{-2\pi\ell\sqrt{\lambda_n}}).$$

Hence from Propositions 6.2 and 6.4 the determinant $\operatorname{Det}\Delta_{\mathbf{M}\times S^1}$ is of the form

$$\operatorname{Det}\Delta_{\mathbf{M}\times S^1} = 4\pi^2\ell^2\mathbf{C}_{\mathbf{M}}\prod_{n=1}^{\infty}|(1-\mathrm{e}^{-2\pi\ell\sqrt{\lambda_n}})|^2, \tag{7.3}$$

where the constant $\mathbf{C}_{\mathbf{M}}$ is given by:

(a) when $\dim\mathbf{M}$ is even, then

$$\mathbf{C}_{\mathbf{M}} = \mathrm{e}^{2\pi\ell\cdot\zeta_{\mathbf{M}}(-1/2)},$$

(b) when $\dim \mathbf{M} = M$ is odd, then

$$\log \mathbf{C_M} = -\sqrt{\pi}\ell \left\{ \lim_{s\to 0} \left(2\sqrt{\pi} \cdot \zeta_{\mathbf{M}} \left(s - \frac{1}{2} \right) + \frac{\mathbf{a}_{(M+1)/2}}{(4\pi)^{(M+1)/2}} \frac{1}{s} \right) + \frac{\Gamma'(1)}{(4\pi)^{M/2}} \mathbf{a}_{(M+1)/2} \right\}. \qquad \square$$

Example 7.2. As an application of our formula (7.1) we give an expression of $\operatorname{Det} \Delta_{S^2\times S^1}$.

For the standard two-dimensional sphere S^2 the spectral zeta-function is

$$\zeta_{S^2}(s) = \sum_{k=1}^{\infty} \frac{2k+1}{k^s(k+1)^s},$$

which converges for $\mathfrak{Re}(s) > 1$. We rewrite this as

$$\begin{aligned} \zeta_{S^2}(s) &= \frac{1}{2^s} + \sum_{k=2}^{\infty} \frac{1}{k^{2s-1}} \left\{ \left(1+\frac{1}{k}\right)^{-s} + \left(1-\frac{1}{k}\right)^{-s} \right\} \\ &= \frac{1}{2^s} + \sum_{k=2}^{\infty} \frac{1}{k^{2s-1}} \sum_{m=0}^{\infty} 2d_{2m}(-s) \left(\frac{1}{k}\right)^{2m} \\ &= \frac{1}{2^s} + 2 \sum_{m=0}^{2m\le n} d_{2m}(-s)(\zeta(2s-1+2m)-1) \\ &\quad + 2 \sum_{2m>n}^{\infty} \sum_{k=2}^{\infty} d_{2m}(-s) \frac{1}{k^{2m-n}} \frac{1}{k^{2s-1+n}}, \end{aligned} \tag{7.4}$$

where we used the expansion $(1+z)^\alpha = \sum d_m(\alpha) z^m$ for $|z| < 1$. Note that for $\alpha > 0$ this series converges for $-1 \le z \le 1$. Then for $\mathfrak{Re}(s) > (2-n)/2$, by the estimate

$$\sum_{2m>n}^{\infty} \sum_{k=2}^{\infty} \left| d_{2m}(-s) \frac{1}{k^{2m-n}} \frac{1}{k^{2s-1+n}} \right| \le \sum_{2m>n}^{\infty} |d_{2m}(-s)| \frac{1}{2^{2m-n}} \sum_{k=2}^{\infty} \frac{1}{k^{2\mathfrak{Re}(s)-1+n}}$$

and the functional relation for the Riemann zeta-function, the expression (7.4) gives us the analytic continuation of $\zeta_{S^2}(s)$ to the complex plane of $\mathfrak{Re}(s) > (2-n)/2$ for each $n > 0$. So we can put $s = -1/2$ in (6.5) and we have an expression of $\zeta_{S^2}(-1/2)$:

$$\begin{aligned} \zeta_{S^2}(-1/2) &= \sqrt{2} + 2\{\zeta(-2)-1\} + 2d_2\left(\frac{1}{2}\right)\{\zeta(0)-1\} \\ &\quad + 2\sum_{m=3}^{\infty} d_{2m}\left(\frac{1}{2}\right)(\zeta(2m-2)-1) = \sqrt{2} - 2\sum_{m=0}^{\infty} d_{2m}\left(\frac{1}{2}\right) \\ &\quad + 2\sum_{m=0}^{\infty} d_{2m}\left(\frac{1}{2}\right)(\zeta(2m-2)) \\ &= -\sum_{m=0}^{\infty} \frac{(4m)!}{2^{4m-1}(4m-1)((2m)!)^2} \zeta(2m-2). \end{aligned}$$

This is also expressed as

$$\zeta_{S^2}\left(-\frac{1}{2}\right) = \frac{4}{9\pi}\int_0^\infty \int_0^\infty \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial y^2}\left(\frac{x+y}{\mathrm{e}^{x+y}-1}\right)\mathrm{e}^{-x}\right)\frac{1}{\sqrt{xy}}\,\mathrm{d}x\,\mathrm{d}y. \tag{7.5}$$

So, finally we have

$$\begin{aligned}\mathrm{Det}\,\Delta_{S^2\times S^1} &= 4\pi^2\ell^2 \prod_{m=1}^{\infty} \mathrm{e}^{-\pi\ell((4m)!/2^{4m-1}(4m-1)((2m)!)^2)\zeta(2m-2)} \\ &\quad\times \prod_{k=1}^{\infty} |(1-\mathrm{e}^{-2\pi\ell\sqrt{k(k+1)}})^{2k+1}|^2.\end{aligned} \tag{7.6}$$

Again by applying Propositions 6.3 and 6.4 we have an alternative representation of the determinant $\mathrm{Det}\,\Delta_{\mathbf{M}\times S^1}$.

Corollary 7.3. *When* $\dim \mathbf{M}$ *is odd, then*

$$\begin{aligned}&\mathrm{Det}\,\Delta_{\mathbf{M}\times S^1} \\ &\quad= \prod_{k=1}^{\infty} \mathrm{e}^{-2\int_0^\infty \{k_{\mathbf{M}}((\ell^2/k^2)x)-(k^2/4\pi\ell^2 x)^{M/2}\sum_{j=0}^{(M+1)/2}\mathbf{a}_j((\ell^2/k^2)x)^j\}\,\mathrm{e}^{-t}t^{-1}\,\mathrm{d}t} \\ &\quad\times \prod_{i=0}^{[M/2]} \mathrm{e}^{-(2/(4\pi\ell^2)^{M/2})\cdot\mathbf{a}_i\cdot\ell^{2i}\cdot\Gamma(i-M/2)\cdot\zeta(2i-M)} \\ &\quad\times \mathrm{e}^{-(2/(4\pi\ell^2)^{M/2})\mathbf{a}_{[(M+1)/2]}\cdot\ell^{M+1}\{\sqrt{\pi}(\log\ell-\mathbf{C_e}/2+1/2\Gamma'(1/2))\}}\,\mathrm{Det}\,\Delta_{\mathbf{M}}.\end{aligned} \tag{7.7}$$

Here we used the formula $\zeta(s) = 1/(s-1) + \mathbf{C_e}s + \mathrm{O}((s-1)^2)$.

When $\dim \mathbf{M}$ *is even, then*

$$\begin{aligned}\mathrm{Det}\,\Delta_{\mathbf{M}\times S^1} &= \prod_{k=1}^{\infty} \mathrm{e}^{-2\int_0^\infty \{k_{\mathbf{M}}((\ell^2/k^2)x)-(k^2/4\pi\ell^2 x)^{M/2}\sum_{j=0}^{(M+1)/2}\mathbf{a}_j((\ell^2/k^2)x)^j\}\,\mathrm{e}^{-t}t^{-1}\,\mathrm{d}t} \\ &\quad\times \prod_{i=0}^{M/2-1} \mathrm{e}^{-(4\mathbf{a}_i\cdot\ell^{2i}/(4\pi\ell^2)^{M/2})((-1)^{M/2-i}/(M/2-i)!)\cdot\zeta'(2i-M)}\,2\pi\ell\,\mathrm{Det}\,\Delta_{\mathbf{M}}.\end{aligned} \tag{7.8}$$

Note that $\zeta(-2k) = 0$ *for* $k = 1, 2, \ldots$.

Remark 7.4. Our formula (7.1) is of course a special case of formulas given in the paper [7] for more general elliptic operators on the product manifolds $\mathbf{M}\times S^1$. However, here we gave an expression of the constant $\mathbf{C_M}$, although the formula itself is not a computable form, especially for $\mathbf{M}$ being odd dimensional. To obtain a further information we must specify the manifolds $\mathbf{M}$. So in the next section we give a more precise form of this factor $\mathbf{C_M}$ for some flat tori.

8. Flat tori

In the last two sections we considere the zeta-regularized determinant for manifolds of a product form as a Riemannian manifold. In this section we deal with the case that the manifolds are two-, three- and four-dimensional flat tori, which are not always of a product form of lower dimensional tori as Riemannian manifolds.

We know by a similar calculation as we showed in Sections 3 and 5 that the zeta-regularized determinant of $(2n+1)$-dimensional Heisenberg manifolds are always of the product form with a factor which is the zeta-regularized determinant of a $2n$-dimensional torus. So of course it is required to determine the zeta-regularized determinant of flat tori to complete the calculation for Heisenberg manifolds. In this section we give an expression of it for two-, three- and four-dimensional flat tori. Although our expressions are not of a computable form within a finite step, the expression for two-dimensional cases are given by the famous limit formula of Kronecker as we cited in Section 3, and higher cases correspond to a generalization of this limit formula. The structure of a generalization was already stated and discussed focusing in their functional relations in the papers [2,3] for more general Dirichlet series than Epstein zeta-functions which are of our cases. Here we treat with the typical Epstein zeta-functions of two, three and four variables. Since it is enough for our purpose to give an explicit analytic continuation of the functions from a left half region in the complex plane to a region including zero, we give them based on the Jacobi identity and the Mellin transformation in a quite elementary way. For this purpose we fix the flat tori in the following way.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be the standard orthonormal basis on $\mathbb{R}^4$ and we fix a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ of the following form:

$$\mathbf{u}_1 = \mathbf{e}_1, \qquad \mathbf{u}_2 = a_{1,2}\mathbf{e}_1 + a_{2,2}\mathbf{e}_2, \quad \text{we put this} = A\mathbf{e}_1 + B\mathbf{e}_2,$$
$$\mathbf{u}_3 = a_{1,3}\mathbf{e}_1 + a_{2,3}\mathbf{e}_2 + a_{3,3}\mathbf{e}_3, \qquad \mathbf{u}_4 = a_{1,4}\mathbf{e}_1 + a_{2,4}\mathbf{e}_2 + a_{3,4}\mathbf{e}_3 + a_{4,4}\mathbf{e}_4,$$
$$a_{2,2}, a_{3,3}, a_{4,4} > 0.$$

(a) $T^2_L \cong \mathbb{R}^2/L$, where

$$L = L_2 = \{n\mathbf{u}_1 + m\mathbf{u}_2 = (n + ma_{1,2}, ma_{2,2}) | n, m \in \mathbb{Z}\}$$
$$= [\{\mathbf{u}_1, \mathbf{u}_2\}] \text{ is a lattice in } \mathbb{R}^2,$$

(b) $T^3 \cong \mathbb{R}^3/L_3$, $L_3 = \{n\mathbf{u}_1 + m\mathbf{u}_2 + l\mathbf{u}_3 | n, m, l \in \mathbb{Z}\} = [\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}]$,
(c) $T^4 \cong \mathbb{R}^4/L_4$, $L_4 = \{n\mathbf{u}_1 + m\mathbf{u}_2 + l\mathbf{u}_3 + k\mathbf{u}_4 | n, m, l, k \in \mathbb{Z}\} = [\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}]$,

of two, three and four dimensions. All flat tori of such dimensions reduce to these cases.

8.1. Kronecker's second limit formula and two-dimensional tori

Since the dual lattice L^* of L is given by

$$L^* = \left\{ \left(n, \frac{m - nA}{B}\right) \in \mathbb{R}^2 \middle| n, m \in \mathbb{Z} \right\},$$

non-zero eigenvalues of the Laplacian on T_L^2 are

$$\left\{ 4\pi^2 \left(n^2 + \frac{(m-nA)^2}{B^2} \right) \middle| n, m \in \mathbb{Z}, (n, m) \neq (0,0) \right\}.$$

The spectral zeta-function $\zeta_{T_L^2}(s)$, is

$$\begin{aligned} &\frac{1}{\Gamma(s)} \int_0^\infty (Z_{T_L^2}(t) - 1) t^{s-1}\, \mathrm{d}t \\ &\quad = \zeta_{T_L^2}(s) = \frac{2}{(4\pi^2)^s} \sum_{n=1}^\infty \sum_{m \in \mathbb{Z}} \frac{1}{(n^2 + ((1/B^2)(m-nA)^2))^s} + \frac{2B^{2s}}{(4\pi^2)^s} \cdot \zeta(2s). \end{aligned} \tag{8.1}$$

By using the Poisson's summation formula we rewrite the first term as follows:

$$\begin{aligned} &\frac{2}{(4\pi^2)^s} \sum_{n=1}^\infty \sum_{m \in \mathbb{Z}} \frac{1}{(n^2 + ((1/B^2)(m-nA)^2))^s} \\ &\quad = \frac{2}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^\infty \sum_{m \in \mathbb{Z}} \frac{1}{n^{2s}} \int_0^\infty \mathrm{e}^{-(1+(1/B^2 n^2)(m-nA)^2)x} x^{s-1}\, \mathrm{d}x \\ &\quad = \frac{2}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^\infty \frac{1}{n^{2s}} \int_0^\infty \sum_{m \in \mathbb{Z}} \sqrt{\frac{\pi B^2 n^2}{x}}\, \mathrm{e}^{-(\pi B n m)^2/x}\, \mathrm{e}^{-2\pi\sqrt{-1} A n m}\, \mathrm{e}^{-x} x^{s-1}\, \mathrm{d}x \\ &\quad = \frac{2}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^\infty \frac{1}{n^{2s}} \int_0^\infty \sum_{m \in \mathbb{Z}, m \neq 0} \sqrt{\frac{\pi B^2 n^2}{x}} \\ &\qquad \times \mathrm{e}^{-(\pi B n m)^2/x}\, \mathrm{e}^{-2\pi\sqrt{-1} A n m}\, \mathrm{e}^{-x} x^{s-1}\, \mathrm{d}x + \frac{2\sqrt{\pi} B \cdot \Gamma(s - 1/2)}{(4\pi^2)^s \cdot \Gamma(s)} \cdot \zeta(2s-1), \end{aligned}$$

so $\zeta_{T_L^2}(s)$ is of the following form.

Proposition 8.1 (Berndt et al. [3,10]).

$$\zeta_{T_L^2}(s) = \mathcal{H}_0(s) + \frac{2\sqrt{\pi} B \cdot \Gamma(s-1/2)}{(4\pi^2)^s \cdot \Gamma(s)} \cdot \zeta(2s-1) + \frac{2B^{2s}}{(4\pi^2)^s} \cdot \zeta(2s),$$

where we put

$$\begin{aligned} \mathcal{H}_0(s) &= \frac{2}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^\infty \frac{1}{n^{2s}} \\ &\quad \times \int_0^\infty \sum_{m \in \mathbb{Z}, m \neq 0} \sqrt{\frac{\pi B^2 n^2}{x}}\, \mathrm{e}^{-(\pi B n m)^2/x}\, \mathrm{e}^{-2\pi\sqrt{-1} A n m}\, \mathrm{e}^{-x} x^{s-1}\, \mathrm{d}x \\ &= \frac{2\sqrt{\pi} B}{(4\pi^2)^s \cdot \Gamma(s)} \sum_{n=1}^\infty \frac{\mathrm{e}^{-2\pi\sqrt{-1} A n m}}{n^{2s}} \int_0^\infty \sum_{m \in \mathbb{Z}, m \neq 0} \mathrm{e}^{-(\pi B n m)^2/x}\, \mathrm{e}^{-x} x^{s-3/2}\, \mathrm{d}x \end{aligned}$$

We know the integrand of $\mathcal{H}_0(s)$ satisfies the asymptotics.

Lemma 8.2. *For any $N \in \mathbb{N}$*

$$\sum_{m\in\mathbb{Z},m\neq 0}\sqrt{\frac{\pi B^2 n^2}{x}}\,\mathrm{e}^{-(\pi Bnm)^2/x}\,\mathrm{e}^{-2\pi\sqrt{-1}Anm}\,\mathrm{e}^{-x}x^{s-1} = \mathrm{O}\left(\frac{x^{N-\mathfrak{Re}(s)-3/2}}{n^{2N-1}}\right). \tag{8.2}$$

Hence the first term $\mathcal{H}_0(s)$ is a holomorphic function of s on the whole complex plane, and

$$\mathcal{H}_0(s) = 2\sqrt{\pi}B\sum_{n=1}^{\infty} n \times \int_0^{\infty}\sum_{m\in\mathbb{Z},m\neq 0}\mathrm{e}^{-(\pi Bnm)^2/x}\,\mathrm{e}^{-2\pi\sqrt{-1}Anm}\,\mathrm{e}^{-x}x^{-1/2-1}\,\mathrm{d}x\cdot s + \mathrm{O}(s^2).$$

Then again by making use of (7.2)

$$\begin{aligned}\mathcal{H}_0'(0) &= 2\sqrt{\pi}B\sum_{n=1}^{\infty} n\,\mathrm{e}^{-2\pi\sqrt{-1}Anm}\int_0^{\infty}\sum_{m\in\mathbb{Z},m\neq 0}\mathrm{e}^{-(\pi Bnm)^2/x}\,\mathrm{e}^{-x}x^{-1/2-1}\,\mathrm{d}x\\ &= \frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z},m\neq 0}\frac{\mathrm{e}^{-2\pi\sqrt{-1}Anm}}{m}\int_0^{\infty}\mathrm{e}^{-(\pi Bnm)^2/x}\,\mathrm{e}^{-x}x^{-1/2}\,\mathrm{d}x\\ &= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}\{\mathrm{e}^{-2\pi nm(B-\sqrt{-1}A)} + \mathrm{e}^{-2\pi nm(B+\sqrt{-1}A)}\}\\ &= -2\sum_{n=1}^{\infty}\{\log(1-\mathrm{e}^{2\pi n(B-\sqrt{-1}A)}) + \log(1-\mathrm{e}^{2\pi n(B+\sqrt{-1}A)})\}.\end{aligned} \tag{8.3}$$

From the facts

$$\zeta(-1) = -\tfrac{1}{12}, \qquad \zeta(s) = -\tfrac{1}{2} - \tfrac{1}{2}s\log 2\pi + \mathrm{O}(s^2)$$

and (7.2) we have

$$\zeta_{T_L^2}(s) = -1 + \left\{\frac{\pi B}{3} - 2\log B - 2\sum_{n=1}^{\infty}\log(1-\mathrm{e}^{-2\pi n(B-\sqrt{-1}A)}) + \log(1-\mathrm{e}^{-2\pi n(B+\sqrt{-1}A)})\right\}\cdot s + \mathrm{O}(s^2).$$

Then the zeta-regularized determinant $\mathrm{Det}\,\Delta_{T_L^2}$ is given by the following formula.

Theorem 8.3.

$$\mathrm{Det}\,\Delta_{T_L^2} = B^2\,\mathrm{e}^{-\pi B/3}\prod_{n=1}^{\infty}|(1-\mathrm{e}^{-2\pi n(B-\sqrt{-1}A)})|^4. \tag{8.4}$$

Corollary 8.4. *From the expression* (8.4) *we can easily see that* Det $\Delta_{T_L^2}$ *is periodic with respect to the parameter* A, *and when* $A = 0$ *we have both of*

$$\lim_{B\to 0} \mathrm{Det}\, \Delta_{T_L^2} = 0, \qquad \lim_{B\to\infty} \mathrm{Det}\, \Delta_{T_L^2} = 0.$$

Now we explain the relation of (8.4) with the Kronecker's second limit formula (see [7] where another explanation is given). By using the integral representation of the modified Bessel function $K_\alpha(z)$ [1]:

$$K_\alpha(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\alpha \int_0^\infty \mathrm{e}^{-t-z^2/4t} t^{-\alpha-1}\,\mathrm{d}t, \quad |\arg z| < \frac{\pi}{4},$$

the function $\mathcal{H}_0(s)$ is expressed as

$$\mathcal{H}_0(s) = \frac{8B^{s+1/2}}{(4\pi)^s \cdot \Gamma(s)} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \cos\,(2\pi\sqrt{-1}Anm)\left(\frac{m}{n}\right)^{s-1/2} K_{1/2-s}(2\pi Bnm). \tag{8.5}$$

Then from this we have a functional relation of the function $\mathcal{H}_0(s)$.

Proposition 8.5.

$$\mathcal{H}_0(1-s) = \frac{(4\pi)^{2s-1} \cdot \Gamma(s)}{\Gamma(1-s) \cdot B^{2s-1}} \mathcal{H}_0(s), \tag{8.6}$$

especially

$$\mathcal{H}_0(1) = \frac{B}{4\pi}\mathcal{H}_0'(0). \tag{8.7}$$

The relation (8.5) together with the functional relation of Riemann zeta-function gives us a very simple functional relation of the spectral zeta-function $\zeta_{T_L^2}(s)$ for two-dimensional flat torus T_L^2.

Corollary 8.6.

$$\Gamma(1-s) \cdot \zeta_{T_L^2}(1-s) = \left(\frac{4\pi}{B}\right)^{2s-1} \cdot \Gamma(s) \cdot \zeta_{T_L^2}(s).$$

From formula (8.1) we can easily see that the function $\zeta_{T_L^2}(s)$ has (only) a pole of order 1 at $s = 1$, which comes from that of the second term in (8.1) and the Kronecker's second limit formula gives the constant term at this pole, that is, by relation (8.7) of the first term $\mathcal{H}_0(s)$ we have the following proposition.

Proposition 8.7 (Kronecker's second limit formula).

$$\lim_{s\to 1}\left\{\zeta_{T_L^2}(s)-\frac{1}{2s-2}\right\}=\mathcal{H}_0(1)+\lim_{s\to 1}\frac{2\sqrt{\pi}B\cdot\Gamma(s-1/2)}{(4\pi)^s\cdot\Gamma(s)}\left\{\zeta(2s-1)-\frac{1}{2s-2}\right\}+\frac{2B^2}{4\pi^2}\cdot\zeta(2)=\frac{B}{4\pi}\mathcal{H}_0'(0)+\frac{B}{2}\mathbf{C_e}+\frac{2B^2}{4\pi^2}\cdot\zeta(2).$$

This gives the following corollary.

Corollary 8.8.

$$\log\operatorname{Det}\Delta_{T^2}-\frac{4\pi}{B}\lim_{s\to 1}\left\{\zeta_{T_L^2}(s)-\frac{1}{2s-2}\right\}=2\log B-2\pi\mathbf{C_e}-B\left\{\frac{\pi}{3}+\frac{2}{\pi}\right\}.$$

8.2. Three-dimensional flat torus

Let

$$\mathfrak{A}=\begin{pmatrix}1 & a_{1,2} & a_{1,3}\\ 0 & a_{2,2} & a_{2,3}\\ 0 & 0 & a_{3,3}\end{pmatrix}$$

and $\mathfrak{G}={}^t\mathfrak{A}^{-1}=(g_{i,j})$, then the dual lattice of L_3 is generated by the basis $\{\mathbf{u}_1^*,\mathbf{u}_2^*,\mathbf{u}_3^*\}$, where $\mathbf{u}_j^*=\sum_i g_{i,j}\mathbf{e}_i^*$ and we know

$$\operatorname{Spec}(\Delta_{T_L^3})=\{4\pi^2((lg_{3,3}+mg_{3,2}+ng_{3,1})^2+(mg_{2,2}+ng_{2,1})^2+(ng_{1,1})^2)|n,m,l\in\mathbb{Z}\}.$$

Then the spectral zeta-function $\zeta_{T_L^3}(s)$ is

$$\frac{1}{(4\pi^2)^s}\sum_{\substack{n,m,l\in\mathbb{Z}\\ n^2+m^2+l^2\neq 0}}\frac{1}{((lg_{3,3}+mg_{3,2}+ng_{3,1})^2+(mg_{2,2}+ng_{2,1})^2+(ng_{1,1})^2)^s}.$$

Put $(mg_{2,2}+ng_{2,1})^2+(ng_{1,1})^2=I(n,m)$ and as before we express $\zeta_{T_L^3}(s)$ as

$$\zeta_{T_L^3}(s)=\frac{1}{(4\pi^2)^s}\frac{1}{\Gamma(s)}\sum_{\substack{n,m,\in\mathbb{Z}\\ n^2+m^2\neq 0}}\frac{1}{I(n,m)^s}\times\sum_{l\in\mathbb{Z}}\int_0^\infty \mathrm{e}^{(-1+(lg_{3,3}+mg_{3,2}+ng_{3,1})^2/I(n,m))x}x^{s-1}\,\mathrm{d}x+\frac{2}{(2\pi g_{3,3})^{2s}}\cdot\zeta(2s).$$

Then this is equal to the expression:

$$\begin{aligned}
&\frac{1}{(4\pi^2)^s}\frac{1}{\Gamma(s)}\sum_{\substack{n,m,\in\mathbb{Z}\\ n^2+m^2\neq 0}}\frac{1}{I(n,m)^s}\\
&\quad\times\int_0^\infty\sum_{l\in\mathbb{Z}}\mathrm{e}^{-(l+m(g_{3,2}/g_{3,3})+n(g_{3,1}/g_{3,3}))^2(g_{3,3}^2/I(n,m))x}\,\mathrm{e}^{-x}x^{s-1}\,\mathrm{d}x\\
&\quad+\frac{2}{(2\pi g_{3,3})^{2s}}\cdot\zeta(2s)\\
&=\frac{1}{(4\pi^2)^s}\frac{1}{\Gamma(s)}\sum_{\substack{n,m,\in\mathbb{Z}\\ n^2+m^2\neq 0}}\frac{1}{I(n,m)^s}\int_0^\infty\sqrt{\frac{\pi I(n,m)}{g_{3,3}^2x}}\\
&\quad\times\sum_{l\in\mathbb{Z}}\mathrm{e}^{-\pi^2I(n,m)l^2/g_{3,3}^2x}\,\mathrm{e}^{2\pi\sqrt{-1}l(m(g_{3,2}/g_{3,3})+n(g_{3,1}/g_{3,3}))}\,\mathrm{e}^{-x}x^{s-1}\,\mathrm{d}x\\
&\quad+\frac{2}{(2\pi g_{3,3})^{2s}}\cdot\zeta(2s)=\frac{\sqrt{\pi}}{g_{3,3}(4\pi^2)^s}\frac{1}{\Gamma(s)}\sum_{\substack{n,m,\in\mathbb{Z}\\ n^2+m^2\neq 0}}\frac{1}{I(n,m)^{s-1/2}}\\
&\quad\times\sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}}\mathrm{e}^{2\pi\sqrt{-1}l(m(g_{3,2}/g_{3,3})+n(g_{3,1}/g_{3,3}))}\int_0^\infty\mathrm{e}^{-\pi^2I(n,m)l^2/g_{3,3}^2x}\,\mathrm{e}^{-x}x^{s-3/2}\,\mathrm{d}x\\
&\quad+\frac{\sqrt{\pi}}{g_{3,3}(4\pi^2)^s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\sum_{\substack{n,m\in\mathbb{Z}\\ n^2+m^2\neq 0}}\frac{1}{I(n,m)^{s-1/2}}+\frac{2}{(2\pi g_{3,3})^{2s}}\cdot\zeta(2s).
\end{aligned}$$

We put this as $\mathcal{A}_0(s)+\mathcal{A}_1(s)+\mathcal{A}_2(s)$, and calculate each $\mathcal{A}_i'(0)$:

(a)

$$\begin{aligned}
&\mathcal{A}_0(0)=0,\\
&\mathcal{A}_0'(s)=-\sum_{\substack{n,m\in\mathbb{Z}\\ n^2+m^2\neq 0}}\log(1-\mathrm{e}^{-2\pi(\sqrt{I(n,m)}/g_{3,3})+2\pi\sqrt{-1}(m(g_{3,2}/g_{3,3})+n(g_{3,1}/g_{3,3}))})\\
&\qquad-\sum_{\substack{n,m\in\mathbb{Z}\\ n^2+m^2\neq 0}}\log(1-\mathrm{e}^{-2\pi(\sqrt{I(n,m)}/g_{3,3})-2\pi\sqrt{-1}(m(g_{3,2}/g_{3,3})+n(g_{3,1}/g_{3,3}))})\\
&\qquad=-2\sum_{n=1}^\infty\sum_{m=1}^\infty\log(1-\mathrm{e}^{-2\pi(\sqrt{I(n,m)}/g_{3,3})\pm2\pi\sqrt{-1}(m(g_{3,2}/g_{3,3})+n(g_{3,1}/g_{3,3}))})\\
&\qquad\quad-2\sum_{n=1}^\infty\sum_{m=1}^\infty\log(1-\mathrm{e}^{-2\pi(\sqrt{I(n,-m)}/g_{3,3})\pm2\pi\sqrt{-1}(-m(g_{3,2}/g_{3,3})+n(g_{3,1}/g_{3,3}))})
\end{aligned}$$

$$-2\sum_{n=1}^{\infty}\log(1-\mathrm{e}^{-2\pi n\{(\sqrt{g_{2,1}^2+g_{1,1}^2}/g_{3,3})\pm\sqrt{-1}(g_{3,1}/g_{3,3})\}})$$
$$-2\sum_{m=1}^{\infty}\log(1-\mathrm{e}^{-2\pi m\{(g_{2,2}/g_{3,3})\pm\sqrt{-1}(g_{3,2}/g_{3,3})\}}).$$

(b)
$$\mathcal{A}_1(s)=\frac{\sqrt{\pi}}{g_{3,3}(4\pi^2)^s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\sum_{\substack{n,m\in\mathbb{Z}\\ n^2+m^2\neq 0}}\frac{1}{I(n,m)^{s-1/2}}$$
$$=\frac{1}{2\sqrt{\pi}g_{1,1}^{2s-1}g_{3,3}}\frac{\Gamma(s+1/2)}{\Gamma(s+1)}\frac{s}{s-1/2}\cdot\zeta_{T_L^2}\left(s-\frac{1}{2}\right),$$

where $\zeta_{T_L^2}(s)$ is the spectral zeta-function of two-dimensional torus T_L^2 corresponding to the lattice $L=L_{A,B}$, $A=-g_{2,1}/g_{2,2}$, $B=g_{1,1}/g_{2,2}$. Then

$$\mathcal{A}_1(0)=0,\qquad \mathcal{A}_1'(0)=-\frac{g_{1,1}}{g_{3,3}}\cdot\zeta_{T_L^2}\left(-\frac{1}{2}\right).$$

Next we express the value $\zeta_{T_L^2}(-1/2)$ in terms of a modified Bessel function $K_\alpha(z)$ for $\alpha=1$:

$$K_\alpha(z)=\frac{1}{2}\left(\frac{z}{2}\right)^\alpha\int_0^\infty \mathrm{e}^{-t-z^2/4t}t^{-\alpha-1}\,\mathrm{d}t,\quad |\arg z|<\frac{\pi}{4}.$$

So we return to the expression (8.1):

$$\zeta_{T_L^2}(s)=\mathcal{H}_0(s)+\frac{2\sqrt{\pi}B\cdot\Gamma(s-1/2)}{(4\pi^2)^s\cdot\Gamma(s)}\cdot\zeta(2s-1)+\frac{2B^{2s}}{(4\pi^2)^s}\cdot\zeta(2s).$$

Then

$$\mathcal{H}_0\left(-\frac{1}{2}\right)=-2\pi B\sum_{n=1}^{\infty}n^2\int_0^\infty\sum_{m\in\mathbb{Z},m\neq 0}\mathrm{e}^{-(\pi Bnm)^2/x}\,\mathrm{e}^{-2\pi\sqrt{-1}Anm}\,\mathrm{e}^{-x}x^{-2}\,\mathrm{d}x$$
$$=-8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n}{m}\right)K_1(2\pi Bnm)\cos(2\pi Anm),\ \frac{2\sqrt{\pi}B\cdot\Gamma(s-1/2)}{(4\pi^2)^s\cdot\Gamma(s)}$$
$$\cdot\zeta(2s-1)_{|s=-1/2}=-\frac{2B}{\pi}\cdot\zeta(3),\ \frac{2B^{2s}}{(4\pi^2)^s}\cdot\zeta(2s)_{|s=-1/2}=-\frac{\pi}{3B}.$$

Hence

$$\mathcal{A}_1'(0)=\frac{g_{1,1}}{g_{3,3}}\left\{8\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n}{m}K_1(2\pi Bnm)\cos(2\pi Anm)+\frac{2B}{\pi}\cdot\zeta(3)+\frac{\pi}{3B}\right\}.$$

(c)
$$\mathcal{A}_2(0)=-1,\qquad \mathcal{A}_2'(0)=2\log g_{3,3}.$$

Finally, for the lattice $L_3=[\{\mathbf{u}_i\}_{i=1}^3]$ we have an expression of the zeta-regularized determinant.

Theorem 8.9.

$$\begin{aligned}
\mathrm{Det}\,\Delta_{T_L^3} &= \prod_{n=1}^{\infty}\prod_{m=1}^{\infty} |1 - \mathrm{e}^{-2\pi\{(\sqrt{I(n,m)}/g_{3,3})+\sqrt{-1}(mg_{3,2}+ng_{3,1})/g_{3,3}\}}|^4 \\
&\quad\times \prod_{n=1}^{\infty}\prod_{m=1}^{\infty} |1 - \mathrm{e}^{-2\pi\{(\sqrt{I(n,-m)}/g_{3,3})+\sqrt{-1}-(mg_{3,2}+ng_{3,1})/g_{3,3}\}}|^4 \\
&\quad\times \prod_{n=1}^{\infty} \left|1 - \mathrm{e}^{-2\pi n\left\{\left(\sqrt{g_{2,1}^2+g_{1,1}^2}/g_{3,3}\right)+\sqrt{-1}(g_{3,1}/g_{3,3})\right\}}\right|^4 \\
&\quad\times \prod_{m=1}^{\infty} |1 - \mathrm{e}^{-2\pi m\{(g_{2,2}/g_{3,3})+\sqrt{-1}(g_{3,2}/g_{3,3})\}}|^4 \\
&\quad\times \mathrm{e}^{-(g_{1,1}/g_{3,3})\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(n/m)K_1(2\pi(g_{1,1}/g_{2,2})nm)\cos(2\pi(g_{2,1}/g_{2,2})nm)\}} \\
&\quad\times \mathrm{e}^{-(2/\pi)(g_{1,1}^2/g_{3,3}g_{2,2})\cdot\zeta(3)}\,\mathrm{e}^{-(\pi/3)(g_{2,2}/g_{3,3})}\left(\frac{1}{g_{3,3}}\right)^2.
\end{aligned}$$

Remark 8.10. As a special case of T^3, let us assume $a_{1,3} = a_{2,3} = 0$. Then the formula of $\mathrm{Det}\,\Delta_{T^3}$ for this case coincides with the formula (7.1) for $T^2 \times S^1$.

8.3. *Four-dimensional flat torus*

Here we only state a formula for a four-dimensional torus and a special case of them. Let

$$\mathfrak{A} = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix}$$

and $\mathbf{u}_j = \sum_i a_{i,j}\mathbf{e}_i$ as explained in the beginning of this section and $\mathfrak{G} = {}^t\mathfrak{A}^{-1} = (g_{i,j})$.

Let $L = L(\mathfrak{A})$ be the lattice generated by $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$, then the dual lattice of $L(\mathfrak{A})$ is generated by the basis $\{\mathbf{u}_1^*, \mathbf{u}_2^*, \mathbf{u}_3^*, \mathbf{u}_4^*\}$, where $\mathbf{u}_j^* = \sum_i g_{i,j}\mathbf{e}_i^*$.

Put

$$(lg_{3,3} + mg_{3,2} + ng_{3,1})^2 + (mg_{2,2} + ng_{2,1})^2 + (ng_{1,1})^2 = I(n, m, l),$$

and

$$\frac{ng_{4,1} + mg_{4,2} + lg_{4,3}}{g_{4,4}} = \alpha(n, m, l)$$

then the spectral zeta-function $\zeta_{T^4_{L(\mathfrak{A})}}(s)$ for this case is written as

$$\begin{aligned}\zeta_{T^4_{L(\mathfrak{A})}}(s) &= \frac{1}{(4\pi^2)^s}\sum_{\substack{n,m,l\in\mathbb{Z}\\ n^2+m^2+l^2\neq 0}}\frac{1}{(I(n,m,l)+(ng_{4,1}+mg_{4,2}+lg_{4,3}+kg_{4,4})^2)^s}\\ &= \frac{1}{(4\pi^2)^s}\frac{1}{\Gamma(s)}\sum_{\substack{n,m,l\in\mathbb{Z}\\ n^2+m^2+l^2\neq 0}}\frac{1}{I(n,m,l)^s}\int_0^\infty\sum_{k\in\mathbb{Z}}\mathrm{e}^{-(k+\alpha(n,m,l))^2(g_{4,4}^2/I(n,m,l))x}\\ &\quad\times \mathrm{e}^{-x}x^{s-1}\,\mathrm{d}x+\frac{2}{(2\pi g_{4,4})^{2s}}\cdot\zeta(2s)=\frac{\sqrt{\pi}}{g_{4,4}(4\pi^2)^s}\frac{1}{\Gamma(s)}\\ &\quad\times\sum_{\substack{n,m,l\in\mathbb{Z}\\ n^2+m^2+l^2\neq 0}}\frac{1}{I(n,m,l)^{s-1/2}}\int_0^\infty\sum_{k\in\mathbb{Z},k\neq 0}\mathrm{e}^{-(\pi^2k^2/xg_{4,4}^2)I(n,m,l)}\\ &\quad\times \mathrm{e}^{2\pi\sqrt{-1}\alpha(n,m,l)k}\,\mathrm{e}^{-x}x^{s-3/2}\,\mathrm{d}x+\frac{\sqrt{\pi}}{g_{4,4}(2\pi)^{2s}}\frac{\Gamma(s-1/2)}{\Gamma(s)}\\ &\quad\times\sum_{\substack{n,m,l\in\mathbb{Z}\\ n^2+m^2+l^2\neq 0}}\frac{1}{I(n,m,l)^{s-1/2}}+\frac{2}{(2\pi g_{4,4})^{2s}}\cdot\zeta(2s).\end{aligned}$$

We put this as $\mathcal{B}_0(s)+\mathcal{B}_1(s)+\mathcal{B}_2(s)$ corresponding to each term.

Note that

$$\mathcal{B}_1(s)=\frac{1}{2\sqrt{\pi}g_{4,4}}\frac{\Gamma(s+1/2)}{\Gamma(s+1)}\frac{s}{s-1/2}\cdot\zeta_{T^3}\left(s-\frac{1}{2}\right),$$

and at $s=-1/2$, $\zeta_{T^3}(s)$ is holomorphic.

Theorem 8.11.

$$\mathrm{Det}\,\Delta_{T^4_{L(\mathfrak{A})}}=\prod_{i=0}^{2}\mathrm{e}^{-\mathcal{B}_i'(0)},$$

where each $\mathcal{B}_i'(0)$ *is given as follows*:

$$\begin{aligned}&\mathcal{B}_0(0)=0,\\ &\mathcal{B}_0'(0)=-\sum_{\substack{n,m,l\in\mathbb{Z}\\ n^2+m^2+l^2\neq 0}}\log(1-\mathrm{e}^{-2\pi\{(\sqrt{I(n,m,l)}/g_{4,4})+\sqrt{-1}\alpha(n,m,l)\}})\\ &\quad-\sum_{\substack{n,m,l\in\mathbb{Z}\\ n^2+m^2+l^2\neq 0}}\log(1-\mathrm{e}^{-2\pi\{(\sqrt{I(n,m,l)}/g_{4,4})-\sqrt{-1}\alpha(n,m,l)\}}),\qquad \mathcal{B}_1(0)=0,\end{aligned}$$

$$\mathcal{B}_1'(0) = -\frac{1}{g_{4,4}} \cdot \zeta_{T^3}\left(-\frac{1}{2}\right) = -\frac{1}{g_{4,4}}\left\{\mathcal{A}_0\left(-\frac{1}{2}\right) + \mathcal{A}_1\left(-\frac{1}{2}\right) + \mathcal{A}_2\left(-\frac{1}{2}\right)\right\}$$

$$= \frac{1}{g_{4,4}}\left\{\sum_{\substack{n,m,\in\mathbb{Z}\\ n^2+m^2\neq 0}}\sum_{l=1}^{\infty} 4\frac{\sqrt{I(n,m)}}{l}\cos\left(2\pi l\frac{mg_{3,2}+ng_{3,1}}{g_{3,3}}\right)\cdot K_1\left(\frac{2\pi}{g_{3,3}l}\sqrt{I(n,m)}\right)\right.$$

$$+ 8\sqrt{\pi}\frac{g_{1,1}^2}{g_{3,3}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n^2}{m}\cos\left(2\pi\sqrt{-1}\frac{g_{2,1}}{g_{2,2}}nm\right)K_1\left(2\pi\frac{g_{1,1}}{g_{2,2}}nm\right)$$

$$\left. + \frac{3g_{1,1}^2}{4\pi^3 g_{3,3}}\cdot\zeta(4) + \frac{2g_{2,2}^2}{\pi g_{3,3}}\cdot\zeta(3) + \frac{\pi}{3}g_{3,3}\right\},$$

$$\mathcal{B}_2(0) = -1, \qquad \mathcal{B}_2'(0) = 2\log g_{4,4}.$$

The formula in Theorem 6.7 and similar one give us several formulas of the zeta-regularized determinant for flat tori defined by matrices of the form

$$\mathfrak{A} = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & 0 \\ 0 & a_{2,2} & a_{2,3} & 0 \\ 0 & 0 & a_{3,3} & 0 \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix}$$

or

$$\mathfrak{A} = \begin{pmatrix} 1 & a_{1,2} & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix}.$$

As an application of the formula (6.7) we state a formula for a torus defined by the latter one, that is, the torus is a direct product of 2 two-dimensional tori as Riemannian manifold.

Let

$$\mathfrak{A} = \begin{pmatrix} 1 & a_{1,2} & 0 & 0 \\ 0 & a_{2,2} & 0 & 0 \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & 0 & a_{4,4} \end{pmatrix}$$

and $L_{\mathfrak{A}}$ the lattice in $\mathbb{R}^4$ generated by $\{\mathbf{u}_1 = \mathbf{e}_1, \mathbf{u}_2 = a_{1,2}\mathbf{e}_1 + a_{2,2}\mathbf{e}_2, \mathbf{u}_3 = a_{3,3}\mathbf{e}_3, \mathbf{u}_4 = a_{3,4}\mathbf{e}_3 + a_{4,4}\mathbf{e}_4\}$. Then the torus $\mathbb{R}^4/L(\mathfrak{A})$ is a direct product of two tori $\mathbf{M}\times\mathbf{N}$, where $\mathbf{M} = \mathbb{R}^2/L_{\mathbf{M}}$, $L_{\mathbf{M}} = [\{\mathbf{u}_1, \mathbf{u}_2\}]$ and $\mathbf{N} = \mathbb{R}^2/L_{\mathbf{N}}$, $L_{\mathbf{N}} = [\{\mathbf{u}_3, \mathbf{u}_3\}]$. Since each heat kernel asymptotics for flat tori vanishes except the first one $= \mathbf{b}_0 =$ volume of the torus, we can rewrite the formula (6.7) for this case as the following corollary.

Corollary 8.12.

$$\mathrm{Det}\,\Delta_{T^4_{L(\mathfrak{A})}} = \mathrm{Det}\,\Delta_{\mathbf{M}} \prod_{\lambda\in L^*_{\mathbf{M}},\lambda\neq 0} \mathrm{e}^{-\int_0^\infty (K_{\mathbf{N}}(t/\|\lambda\|^2)-\|\lambda\|^2\mathbf{b}_0/4\pi t)\,\mathrm{e}^{-t}t^{-1}\,\mathrm{d}t} \times \mathrm{e}^{-\mathbf{b}_0/4\pi\{\lim_{s\to 0}\Gamma(s-1)\cdot\zeta_{\mathbf{M}}(s-1)\}},$$

where

$$\mathrm{Det}\,\Delta_{\mathbf{M}} = B^2\,\mathrm{e}^{-\pi B/3}\prod_{n=1}^{\infty}|(1-\mathrm{e}^{-2\pi n(B-\sqrt{-1}A)})|^4,$$

$$\prod_{\lambda\in L^*_{\mathbf{M}},\lambda\neq 0} \mathrm{e}^{-\int_0^\infty (K_{\mathbf{N}}(t/\|\lambda\|^2)-\|\lambda\|^2\mathbf{b}_0/4\pi t)\,\mathrm{e}^{-t}t^{-1}\,\mathrm{d}t} = \prod_{\lambda\in L^*_{\mathbf{M}},\lambda\neq 0}\ \prod_{\gamma\in L_{\mathbf{N}},\gamma\neq 0} \mathrm{e}^{-(\|\lambda\|\mathbf{b}_0/\pi\|\gamma\|)K_1(\|\lambda\|\|\gamma\|)},$$

and

$$\begin{aligned}
&\lim_{s\to 0}\Gamma(s-1)\cdot\zeta_{\mathbf{M}}(s-1)\\
&= \lim_{s\to -1}\left\{\frac{2}{(4\pi^2)^s}\sum_{n=1}^{\infty}\frac{1}{n^{2s}}\int_0^\infty \sum_{m\in\mathbb{Z},m\neq 0}\sqrt{\frac{\pi B^2 n^2}{x}}\,\mathrm{e}^{-(\pi Bnm)^2/x}\right.\\
&\quad\times \mathrm{e}^{-2\pi\sqrt{-1}Anm}\,\mathrm{e}^{-x}x^{s-1}\,\mathrm{d}x + \frac{2\sqrt{\pi}B}{(4\pi^2)^s}\cdot\Gamma\left(s-\frac{1}{2}\right)\cdot\zeta(2s-1)\\
&\quad\left. + \frac{\Gamma(s)\cdot B^{2s}}{(4\pi^2)^s}\cdot\zeta(2s)\right\}\\
&= \frac{32\pi}{\sqrt{B}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n}{m}\right)^{3/2}\cos(2\pi Anm)K_{3/2}(2\pi Bnm) + \frac{8\pi}{B}\cdot\zeta(4) + \frac{4}{B^2}\cdot\zeta(3)\\
&= \left(\frac{4\pi}{B}\right)^3\Gamma(2)\zeta_{T^2}(2) = 4\pi B\sum_{\substack{n,m\in\mathbb{Z}\\ n^2+m^2\neq 0}}\frac{1}{((Bn)^2+(m-nA)^2)^2},
\end{aligned}$$

here $B = g_{1,1}/g_{2,2}$, $A = -g_{2,1}/g_{2,2}$, $\mathbf{b}_0 = \textit{volume of}\,\mathbf{N} = a_{3,3}a_{4,4}$.

Note that the second term is obtained by making use of the Jacobi identity:

$$\sum_{\mu\in L^*_{\mathbf{N}}}\mathrm{e}^{-t\|\mu\|^2} = \frac{\mathbf{b}_0}{4\pi t}\sum_{\gamma\in L_{\mathbf{N}}}\mathrm{e}^{-\|\gamma\|^2/4t}.$$

Finally, we note that in the most special case, that is, let the matrix $\mathfrak{A}$ be the identity matrix, then we have a formula for the spectral zeta-function $\zeta_{T^4}(s)$

$$(4\pi^2)^{-s}\zeta_{T^4}(s) = 8(1-2^{2-2^s})\zeta(s)\zeta(s-1) \tag{8.8}$$

(and in each dimension we have similar formulas). So by this formula we have simply state the following corollary.

Corollary 8.13. *Let the torus T^4 be defined by the lattice $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, then the zeta-regularized determinant* Det Δ_{T^4} *is given explicitly in the form*

$$\log \mathrm{Det}\, \Delta_{T^4} = 2^4(\log 2\pi + 2(\log 2)^2)\zeta(0)\zeta(-1) - 2^3(\zeta'(0)\zeta(-1) + \zeta(0)\zeta'(-1)).$$

It is possible to simplify this formula by using several formulas of Riemann zeta-function $\zeta(s)$ and to compare with our formula.

Remark 8.14. Similar to the case of $\zeta_{T_L^2}(s)$, the function $\zeta_{T_L^3}(s)$ (respectively, $\zeta_{T_L^4}(s)$) has only a pole at $s = 3/2$ (respectively, $s = 2$) of order 1 coming from the second term $\mathcal{A}_1(s)$ (respectively, $\mathcal{B}_1(s)$) and the term $\mathcal{A}_0(s)$ (resp. $\mathcal{B}_0(s)$) will correspond to the term $\mathcal{H}_0(s)$ in the two-dimensional cases and there are similar functional relations like (8.5) also in these cases which are derived from the Jacobi identity.

References

[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.
[2] B.C. Berndt, Identities involving the coefficients of a class of Dirichlet series V, Trans. Am. Math. Soc. 160 (1971) 139–156.
[3] B.C. Berndt, Identities involving the coefficients of a class of Dirichlet series VI, Trans. Am. Math. Soc. 160 (1971) 157–167.
[4] J. Bolte, F. Steiner, Determinants of Laplace-like operators on Riemann surfaces, Commun. Math. Phys. 130 (1990) 581–597.
[5] E. D'Hoker, D.H. Phong, The geometry of string perturbation theory, Rev. Mod. Phys. 60 (1988) 917–1065.
[6] C. Gordon, E. Wilson, The spectrum of the Laplacian on Riemannian Heisenberg manifolds, Michigan Math. J. 33 (1986) 253–271.
[7] R. Forman, Functional determinants and geometry, Invent. Math. 88 (1987) 447–493.
[8] K. Furutani, The heat kernel and the spectrum of a class of nilmanifolds, Commun. Partial Diff. Eqs. 21 (3-4) (1996) 423–438.
[9] H.P. Mckean, Selberg's trace formula as applied to compact Riemann surfaces, Commun. Pure Appl. Math. 25 (1972) 225–271.
[10] Y. Motohashi, A new proof of the limit formula of Kronecker, Proc. Jpn. Acad. 44 (1968) 614–616.
[11] C. Nash, D.J. O'Connor, Determinants of Laplacians, Ray–Singer torsion on lens spaces and the Riemann zeta function, J. Math. Phys. 36 (3) (1995) 1462–1505.
[12] J.R. Quine, S.H. Heydari, R.Y. Song, Zeta regularized products, Trans. Am. Math. Soc. 338 (1) (1993) 213–231.
[13] D.B. Ray, I.M. Singer, R-torsion and the Laplacian on Riemannian manifolds, Adv. Math. 7 (1971) 145–210.
[14] I. Vardi, Determinants of Laplacians and multiple gamma functions, SIAM J. Math. Anal. 19 (1988) 493–507.